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PHD

## Exchange graphs and stability conditions for quivers

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# Exchange graphs and stability conditions for quivers

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

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Yu Qiu

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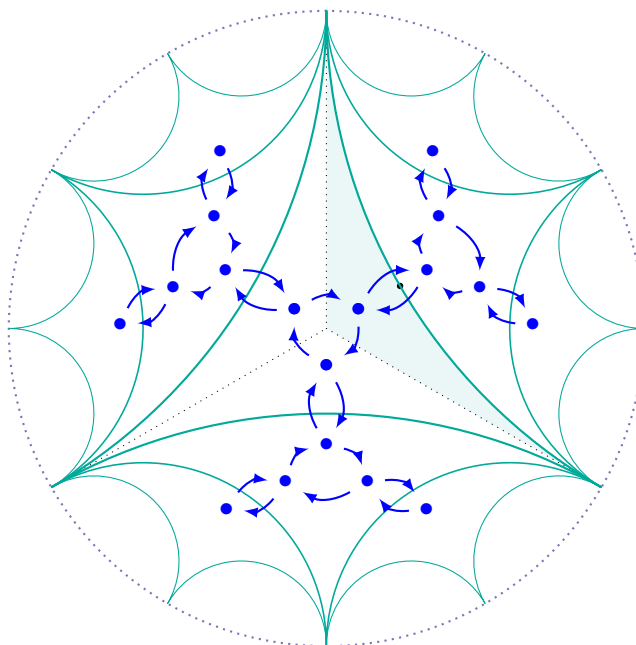
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## 1.1 General context

The notion of a stability condition on a triangulated category was defined by Bridgeland [5] (c.f. Section 2.9). The idea was inspired from physics by studying D-branes in string theory. Nevertheless, the notion itself is interesting purely mathematically. A stability condition on a triangulated category  $\mathcal{D}$  consists of a collection of full additive subcategories of  $\mathcal{D}$ , known as the slicing, and a group homomorphism from the Grothendieck group  $\mathcal{K}(\mathcal{D})$  to the complex plane, known as the central charge. Bridgeland [5] showed a key result that the space  $\text{Stab}(\mathcal{D})$  of all stability conditions on  $\mathcal{D}$  is a finite dimensional complex manifold. Moreover, these spaces carry interesting geometric/topological structure which shade light on the properties of the original triangulated categories. Most interesting examples of triangulated categories are derived categories. They are weak homological invariants arising in both algebraic geometry and representation theory, and indeed different manifolds and quivers (usually with relation) might share the same derived category (say complex projective line and Kronecker quiver). Also note that the stability spaces are closely related to the mirror categories of these derived categories, in the sense of Kontsevich's homological mirror symmetry, i.e. the Fukaya categories of Lagrangian submanifolds of certain symplectic manifolds. Usually, because of physical motivation, people (c.f. [6]) are primarily interested in the stability spaces of the derived categories of coherent sheaves on Calabi-Yau 3-folds (and 2-folds). However we will study the stability space of the bounded derived category  $\mathcal{D}(Q)$  of a quiver  $Q$ .

In understanding stability conditions and triangulated categories, (bounded) t-structures play an important role. In fact, we can view a t-structure as a ‘discrete’ (integer) structure while a stability condition (resp. a slicing) is its ‘complex’ (resp. ‘real’) refinement. Every t-structure carries an abelian category sitting inside it, known



**Figure 1-1:** *Exchange graphs as the skeleton of stability spaces*

as its heart. Note that an abelian category is a canonical heart in its derived category, e.g.  $\mathcal{H}_Q = \text{mod } \mathbf{k}Q$  is the canonical heart of  $\mathcal{D}(Q)$ . The classical way to understand relations between different hearts is via HRS-tilting (c.f. Section 2.3), in the sense of Happel-Reiten-Smalø. Note that to give a stability condition is equivalent to giving a t-structure and a stability function on its heart with the Harder-Narashimhan (HN) property. This implies that a finite heart (i.e. has  $n$  simples and has finite length) corresponds to a (complex)  $n$ -cell in the stability space. Moreover, Woolf [51] shows that the tilting between finite hearts corresponds to the tiling of these  $n$ -cells. More precisely, two  $n$ -cells meet if and only if the corresponding hearts differ by a HRS-tilting; and they meet in codimension one if and only if the hearts differ by a simple tilting. Thus, our main method to study a stability space of a triangulated category  $\mathcal{D}$  is via its ‘skeleton’ – the exchange graph  $\text{EG}(\mathcal{D})$ , that is, the oriented graphs whose vertices are hearts in  $\mathcal{D}$  and whose edges correspond to simple (forward) tiltings between them. Figure 1-1 demonstrates the duality between the exchange graph and the tiling of the stability space by many cells like the shaded area, so that each vertex in the exchange graph corresponds to a cell and each edge corresponds to a common edge (codimension one face) of two neighboring cells.

Exchange graphs also appear in cluster theory, which has many links to various areas in mathematics (see the survey [29]). Cluster algebras (together with quiver mutation, c.f. Section 2.5) were invented by Fomin-Zelevinsky [16] in 2000 whose original motivation is to understand the total positivity and canonical bases. The cluster exchange graph is the unoriented graph whose vertices are clusters consisting of variables and



whose edges are mutations between the quivers associated to the clusters. Moreover, such graphs only depend on the mutation equivalent class of the associated quivers. In the case that there is an acyclic quiver  $Q$  in such class, Buan-Marsh-Reineke-Reiten-Todorov [9] categorified the corresponding cluster algebras and introduced (normal) cluster categories via  $\mathcal{D}(Q)$ . Keller [31] showed that one can construct a more general  $m$ -cluster category  $\mathcal{C}_m(Q)$  ( $m \geq 1$ ) as the orbit category  $\mathcal{D}(Q)/\Sigma_m$  for certain auto-equivalence  $\Sigma_m \in \text{Aut } \mathcal{D}(Q)$  (see Section 2.6). The corresponding exchange graph  $\text{CEG}_m(Q)$  of  $m$ -clusters is the oriented graph whose vertices are  $m$ -cluster tilting sets and whose edges are mutations. Note that there is a cyclic structure ( $m$ -cycles) in  $\text{CEG}_m(Q)$  which corresponds to repeatedly mutating a  $m$ -cluster at the same place; while the original unoriented cluster exchange graph can be obtained from  $\text{CEG}_2(Q)$  by replacing every 2-cycle with a unoriented edge. Keller's construction began to reveal the relation between  $\text{CEG}_m(Q)$  and the principal component  $\text{EG}^\circ(Q)$  of the exchange graph of  $\mathcal{D}(Q)$ , that is, the connected component containing the canonical heart  $\mathcal{H}_Q$ .

Amiot [1] showed that  $(N - 1)$ -cluster category  $\mathcal{C}_{N-1}(Q)$  can be constructed as quotient category via the following ‘short exact sequence’ of triangulated categories

$$0 \rightarrow \mathcal{D}(\Gamma_N Q) \rightarrow \text{per}(\Gamma_N Q) \rightarrow \mathcal{C}_{N-1}(Q) \rightarrow 0, \quad (1.1.1)$$

where  $\mathcal{D}(\Gamma_N Q)$  (resp.  $\text{per}(\Gamma_N Q)$ ) is the finite-dimensional (resp. perfect) derived category of the Calabi-Yau- $N$  Ginzburg algebra  $\Gamma_N(Q)$  associated to  $Q$  (c.f. [22]). This leads to another main class of our examples of triangulated categories. We will study the principal component  $\text{EG}^\circ(\Gamma_N Q)$  of the exchange graph of  $\mathcal{D}(\Gamma_N Q)$ , that is, the connected component containing the canonical heart  $\mathcal{H}_\Gamma$ . One of our main goals is to describe the relationship between the different exchange graphs  $\text{CEG}_{N-1}(Q)$  and  $\text{EG}^\circ(Q)$ , and further, the relationship between  $\text{EG}^\circ(\Gamma_N Q)$  for different values of  $N$ . A philosophical point of view that emerges is that, in a suitable sense,

$$Q = \lim_{N \rightarrow \infty} \Gamma_N Q. \quad (1.1.2)$$

On the other hand,  $N = 2$  and  $N = 3$  are special cases, in that  $\text{EG}^\circ(\Gamma_2 Q)$  is almost trivial and  $\text{EG}^\circ(\Gamma_3 Q)$  behaves nicely not only because it corresponds to the original cluster algebras, but also because it is more uniform.

Stability conditions naturally link to Donaldson-Thomas (DT) theory, which provides an interpretation of quantum dilogarithm identities. Reineke [48] (c.f. Section 6.6.1) realized the DT-invariant for a Dynkin quiver can be calculated as a product of quantum dilogarithms, indexing by any HN-stratum of  $\mathcal{H}_Q$ , which is a ‘maximal refined version’ of torsion pairs on an abelian category. His approach was integrating certain identities in Hall algebras to show the stratum-independence of the product. We will apply exchange graphs to give a combinatorial proof of such quantum diloga-

rithm identities.

## 1.2 Organization

We will collect related background in Chapter 2. The remaining chapters are organized with respect to their generality, that is, whether the quiver should be acyclic, of Dynkin type or of linear type (i.e.  $A_n$ -type).

In the ‘acyclic chapters’, i.e. Chapter 3 and Chapter 5, we will study, for an acyclic quiver  $Q$ , the principal component  $\text{EG}^\circ(Q)$  and  $\text{EG}^\circ(\Gamma_N Q)$  of the exchange graphs. There is a special structure (the lines, see Definition 3.2.1) in the exchange graphs, by tilting the same simple (up to shift). Using this linear structure, we have the crucial technical notion of convexity and cyclic completion (Definition 3.2.3) of subgraphs in the exchange graphs. We are interested in a certain convex subgraph  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  (with base  $\mathcal{H}_Q$ ) in  $\text{EG}^\circ(Q)$  (c.f. Section 3.2), whose cyclic completion  $\overline{\text{EG}}_N^\circ(Q, \mathcal{H}_Q)$  is identified with the exchange graph  $\text{CEG}_{N-1}(Q)$ ; this is an interpretation of a result of Buan-Reiten-Thomas [10].

Next, we construct an isomorphism  $\mathcal{I}_*$  from  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  to a similar convex subgraph  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  (with base  $\mathcal{H}_\Gamma$ ) in  $\text{EG}^\circ(\Gamma_N Q)$ , induced by a canonical functor  $\mathcal{I} : \mathcal{D}(Q) \rightarrow \mathcal{D}(\Gamma_N Q)$  which is an ‘Lagrangian immersion’ (Definition 3.3.1) in the following sense. Considering the heart algebra  $\mathcal{E}_\mathcal{H}$ , that is, the full  $\text{Hom}^\bullet$  algebra of the simples in a heart  $\mathcal{H}$ , as the tangent space of  $\mathcal{H}$ ; then the tangent space  $\mathcal{E}_\mathcal{H}$  of a heart  $\mathcal{H} \in \text{EG}_N(Q, \mathcal{H}_Q)$  is a subspace of the tangent space  $\mathcal{E}_{\mathcal{I}_*(\mathcal{H})}$  of  $\mathcal{I}_*(\mathcal{H})$  while quotient space is the (graded and shifted) dual of  $\mathcal{E}_\mathcal{H}$ . Also note that  $\mathcal{I}_*$  preserves the linear structure and hence induces the isomorphism between the corresponding cyclic completions.

The interesting fact about  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  is that, it is a fundamental domain for  $\text{EG}^\circ(\Gamma_N Q)/\text{Br}$ , where  $\text{Br} = \text{Br}(\Gamma_N Q)$  is a subgroup of the auto-equivalence group  $\text{Aut } \mathcal{D}(\Gamma_N Q)$ , known as the Seidel-Thomas braid group, which is generated by the twist functors (c.f. Section 2.8). Then, we prove the main results of the thesis, i.e. Theorem 5.1.1 and Theorem 5.2.1, which say that there is a commutative diagram of canonical isomorphisms

$$\begin{array}{ccc}
 \overline{\text{EG}}_N^\circ(Q, \mathcal{H}_Q) & \xrightarrow{\overline{\mathcal{J}}} & \text{CEG}_{N-1}(Q) \\
 \downarrow \overline{\mathcal{I}_*} & & \uparrow \overline{v} \\
 \overline{\text{EG}}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma) & \xrightarrow{\overline{p_0}} & \text{EG}^\circ(\Gamma_N Q)/\text{Br}
 \end{array} \tag{1.2.1}$$

Note that, via  $\overline{v}$ , the linear structure (the lines) in  $\text{EG}^\circ(\Gamma_N Q)$  corresponds to the cyclic structure (the  $N - 1$ -cycles) in  $\text{CEG}_{N-1}(Q)$ . Since  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  will cover the whole  $\text{EG}^\circ(Q)$  when  $N$  tends to infinity, a consequence of the isomorphism  $\overline{p_0} \circ \overline{\mathcal{I}_*}$  in (1.2.1)

can be presented as following limit formula

$$\mathrm{EG}^\circ(Q) \cong \lim_{N \rightarrow \infty} \mathrm{EG}^\circ(\Gamma_N Q) / \mathrm{Br}(\Gamma_N Q),$$

which reflects our philosophy (1.1.2). Further, we show the special uniformity of Calabi-Yau-3 case, (c.f. Theorem 5.4.6), that the convex subgraph  $\mathrm{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$  with any base  $\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_N Q)$  is a fundamental domain for  $\mathrm{EG}^\circ(\Gamma_3 Q) / \mathrm{Br}_3$ . This implies that  $\mathrm{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$  is an oriented copy of the original (unoriented) cluster exchange graph and hence  $\mathrm{EG}^\circ(\Gamma_3 Q)$  is covered by many such copies while the gluing structure is shown in Proposition 5.4.5.

In the ‘Dynkin chapters’, i.e. Chapter 4 and Chapter 6, we consider the ‘finite’ case, i.e. when the quiver  $Q$  is of Dynkin type, in the sense that there are finitely many indecomposables in  $\mathcal{H}_Q$ . We will prove additional stronger properties for exchange graphs. For instance, in Section 4.2, we give another proof of connectedness of the whole exchange graph  $\mathrm{EG}(Q)$  (a result proved earlier by Keller-Vossieck [38]). This connectedness also implies that the stability space  $\mathrm{Stab}(\mathcal{D}(Q))$  is connected, which is false if  $Q$  is not Dynkin type. Also, we will show a simple but important fact that,  $\mathrm{EG}(Q)$  is covered by squares and pentagons as in (6.1.4). Consequently, we prove (Theorem 6.1.6) the simply connectedness of the stability space  $\mathrm{Stab}(\mathcal{D}(Q))$ . For the Calabi-Yau case,  $\mathrm{Stab}(\mathcal{D}(\Gamma_N Q))$  is not connected. Nevertheless, we can identify a principal component  $\mathrm{Stab}^\circ(\Gamma_N Q)$  which corresponds to  $\mathrm{EG}^\circ(Q)$  (Section 6.2.1). Moreover, we show that when we embed the skeleton  $\mathrm{EG}^\circ(\Gamma_N Q) / \mathrm{Br}$  into  $\mathrm{Stab}^\circ(\Gamma_N Q) / \mathrm{Br}$ , the image of basic cycles induced by lines in  $\mathrm{EG}^\circ(\Gamma_N Q)$  are generators of  $\pi_1(\mathrm{Stab}^\circ(\Gamma_N Q) / \mathrm{Br})$  (Theorem 6.2.5). Further, the faithfulness of braid group action will implies the simply connectedness of  $\mathrm{Stab}(\mathcal{D}(\Gamma_N Q))$ . We will also show (Theorem 6.3.2) that the limit formula of exchange graphs can be strengthened as

$$\mathrm{Stab}(Q) \cong \lim_{N \rightarrow \infty} \mathrm{Stab}^\circ(\Gamma_N Q) / \mathrm{Br}(\Gamma_N Q).$$

In Section 6.5 and Section 6.6.2, we discuss directed paths in exchange graphs. We will first show (Theorem 6.5.10) that, HN-strata of  $\mathcal{H}_Q$  can be naturally interpreted as directed paths connecting  $\mathcal{H}_Q$  and  $\mathcal{H}_Q[1]$  in  $\mathrm{EG}(Q)$ . Then the existence of DT-invariant of  $Q$ , is equivalent to the path-independence of the quantum dilogarithm product over such directed paths. We give a slight generalization (Theorem 6.6.3) of this path-independence, to all paths (not necessarily directed) whose vertices lie between  $\mathcal{H}_Q$  and  $\mathcal{H}_Q[1]$ . The point is that this path-independence reduces to the cases of squares and pentagons in (6.1.4); therefore such type of quantum dilogarithm identities are just compositions of the classical Pentagon Identities. Keller [32] also spotted this phenomenon and proved a more remarkable quantum dilogarithm identities via mutation of quivers with potential. Besides, we will discuss the wall-crossing formula

for APR-tilting, (c.f. [43]).

In Chapter 7, we will show even stronger properties for  $Q$  is of  $A_n$ -type. First, we classify the hearts in  $\text{EG}(A_n)$  and  $\text{EG}^\circ(\Gamma_N A_n)$  via graded gentle trees/cycles, which generalizes a result of Assem-Happel [2] about the generalized tilted algebras of type  $A_n$ . Second, we explicitly construct the exchange graph of Calabi-Yau  $A_2$ -type via the Farey graph (Section 7.4). Third, we calculate the stability spaces of  $A_2$ -type and Calabi-Yau  $A_2$ -type. These  $A_2$ -type examples illustrate the idea of the thesis, that exchange graphs are skeleton of stability spaces.

### 1.3 Related work

As partially mentioned above, some previous work related to exchange graphs is as follows:

- Keller constructed the orbit categories [31] and discussed various properties of the Calabi-Yau category  $\mathcal{D}(\Gamma_N Q)$  [34]/[37]. He also gave the green mutation formulae for DT-invariants [32];
- Amiot showed that the hearts in  $\text{EG}^\circ(\Gamma_N Q)$  induce  $(N - 1)$ -cluster tilting sets via (1.1.1);
- Buan-Thomas [11] constructed colored quiver mutation, the combinatorial description of higher cluster tilting;
- Buan-Reiten-Thomas [10] proved that there is a bijection between hearts in  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  and  $(N - 1)$ -clusters;
- Keller-Nicolás' (c.f. [32, Theorem 5.6]) has proved the isomorphism  $\bar{v}$  in (1.2.1) in Calabi-Yau-3 case for quivers with potential.

Previous work concerning ‘finite type’ stability spaces includes

- Thomas [50] calculated the stability space  $\text{Stab}^\circ(\Gamma_2 A_n)$ ;
- Bridgeland [7] showed the stability space  $\text{Stab}^\circ(\Gamma_2 Q)$  is a covering space of  $\mathfrak{h}^{reg}/W$  for  $Q$  is of Dynkin or affine Dynkin type.
- Brav and Thomas [4] proved that Bridgeland’s covering of  $\text{Stab}^\circ(\Gamma_2 Q)$  is universal for  $Q$  is CY-2 Dynkin type;
- Okada [45] calculated the stability space of the Kronecker quiver, which is conformally isomorphic to  $\mathbb{C}^2$ ;
- Woolf [51] presented how to identify a connected component of stability spaces for finite type.

Other related work includes

- Khovanov-Seidel-Thomas [40]/[47] studied  $\mathcal{D}(\Gamma_N A_n)$  via the derived Fukaya category of Lagrangian submanifolds of the Milnor fibres of the singularities of type  $A_n$ ; they also proved that there is a faithful braid group action on  $\mathcal{D}(\Gamma_N A_n)$ ;
- Reineke [48] defined DT-invariant of a Dynkin quiver (via Hall algebra);
- Kontsevich-Soibelman [42] defined the *motivic* DT-invariants for quiver with potential via motivic Hall algebra;
- Nagao [43] gave a wall-crossing formula for DT-invariants for quivers.

## CHAPTER 2

## BACKGROUND

In this chapter, we will collect facts in the theory of the following topics: quiver representation, derived category, cluster algebra/category, Calabi-Yau category and stability condition.

### 2.1 Quivers and Dynkin diagrams

A *quiver*  $Q$  is an oriented graph, i.e. consisting of a vertex set  $Q_0$  and an arrow set  $Q_1$  with maps  $h : Q_1 \rightarrow Q_0$  and  $t : Q_1 \rightarrow Q_0$ , which indicate the *head* and *tail* of an arrow. A quiver is *finite* if  $Q_0$  and  $Q_1$  are finite. An *m-cycle*, for an integer  $m \geq 1$ , in a quiver  $Q$  consists of  $m$  arrows  $a_1, \dots, a_m \in Q_1$  such that  $h(a_i) = t(a_{i+1})$ , for  $i = 1, \dots, m-1$ , and  $h(a_m) = t(a_1)$ . A *loop* in a quiver  $Q$  is a 1-cycle. An acyclic quiver is a quiver without  $m$ -cycles, for any  $m \geq 1$ .

We denote by  $\mathbf{k}Q$ , the *path algebra*, which is then finite dimensional if  $Q$  is acyclic; denote by  $\text{mod } \mathbf{k}Q$  the *category of finite dimensional  $\mathbf{k}Q$ -modules*, which can be identified with  $\text{Rep}_{\mathbf{k}}(Q)$ , the *category of representations of  $Q$*  (c.f. [3]). We will not distinguish between  $\text{mod } \mathbf{k}Q$  and  $\text{Rep}_{\mathbf{k}}(Q)$ . Recall that the *Euler form*

$$\langle -, - \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$$

associated to the quiver  $Q$ , is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{(i \rightarrow j) \in Q_1} \alpha_i \beta_j.$$

Moreover for  $M, L \in \text{mod } \mathbf{k}Q$ , we have

$$\langle \dim M, \dim L \rangle = \dim \text{Hom}(M, L) - \dim \text{Ext}^1(M, L), \quad (2.1.1)$$

where  $\dim E \in \mathbb{N}^{Q_0}$  is the *dimension vector* of any  $E \in \text{mod } \mathbf{k}Q$ .

**Definition 2.1.1.** A (*simply laced*) *Dynkin diagram* is one of the following unoriented graphs:

$$\begin{array}{ll}
 A_n : & 1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } n \\
 D_n : & 1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } n-2 \begin{array}{l} \nearrow n-1 \\ \searrow n \end{array} \\
 E_{6,7,8} : & \begin{array}{c} 4 \\ | \\ 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 5 \text{ --- } 6 \text{ --- } 7 \text{ --- } 8 \end{array}
 \end{array} \tag{2.1.2}$$

A *Dynkin quiver* is a quiver  $Q$  whose underlying unoriented graph is a Dynkin diagram.

Recall Gabriel's theorem of the classification of representation-finite quivers.

**Theorem 2.1.2** (Gabriel [20]). *Let  $Q$  be a connected quiver. We have the following equivalent statements:*

- $Q$  is representation-finite;
- The Euler form  $\langle -, - \rangle$  associated to  $Q$  is positive definite;
- $Q$  is a Dynkin quiver.

Moreover, in this case, the map  $\dim : \text{mod } \mathbf{k}Q \rightarrow \mathbb{N}^{Q_0}$  yields a bijection from the set of isomorphism classes of indecomposable modules to the set of positive roots of the Euler form.

## 2.2 Derived categories

Let  $Q$  be an acyclic quiver and let  $\mathcal{D}(Q) = \mathcal{D}^b(\text{mod } \mathbf{k}Q)$  be its *bounded derived category*, which is a triangulated category.

Recall (e.g. from [5]) that a *t-structure* on a triangulated category  $\mathcal{D}$  is a full subcategory  $\mathcal{P} \subset \mathcal{D}$  with  $\mathcal{P}[1] \subset \mathcal{P}$  and such that, if one defines

$$\mathcal{P}^\perp = \{G \in \mathcal{D} : \text{Hom}_{\mathcal{D}}(F, G) = 0, \forall F \in \mathcal{P}\},$$

then, for every object  $E \in \mathcal{D}$ , there is a unique triangle  $F \rightarrow E \rightarrow G \rightarrow F[1]$  in  $\mathcal{D}$  with  $F \in \mathcal{P}$  and  $G \in \mathcal{P}^\perp$ . Any t-structure is closed under sums and summands and hence it is determined by the indecomposables in it. Note also that  $\mathcal{P}^\perp[-1] \subset \mathcal{P}^\perp$ .

A t-structure  $\mathcal{P}$  is *bounded* if

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{P}^\perp[i] \cap \mathcal{P}[j],$$

or equivalently if, for every object  $M$ , the shifts  $M[k]$  are in  $\mathcal{P}$  for  $k \gg 0$  and in  $\mathcal{P}^\perp$  for  $k \ll 0$ . The *heart* of a t-structure  $\mathcal{P}$  is the full subcategory

$$\mathcal{H} = \mathcal{P}^\perp[1] \cap \mathcal{P}$$

and any bounded t-structure is determined by its heart. More precisely, any bounded t-structure  $\mathcal{P}$  with heart  $\mathcal{H}$  determines, for each  $M$  in  $\mathcal{D}$ , a canonical filtration ([5, Lemma 3.2])

$$\begin{array}{ccccccc}
 0 = M_0 & \longrightarrow & M_1 & \longrightarrow & \dots & \longrightarrow & M_{m-1} & \longrightarrow & M_m = M \\
 & & \swarrow & & & & \swarrow & & \\
 & & H_1[k_1] & & & & H_m[k_m] & & 
 \end{array} \tag{2.2.1}$$

where  $H_i \in \mathcal{H}$  and  $k_1 > \dots > k_m$  are integers. Using this filtration, one can define the  $k$ -th homology of  $M$ , with respect to  $\mathcal{H}$ , to be

$$\mathbf{H}_k(M) = \begin{cases} H_i & \text{if } k = k_i \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.2}$$

Then  $\mathcal{P}$  consists of those objects with no nontrivial negative homology and  $\mathcal{P}^\perp$  those with only negative homology.

**Definition 2.2.1.** Let  $\mathcal{D}$  be a triangulated category with heart  $\mathcal{H}$ . For any object  $M$  in  $\mathcal{D}$ , we define the (*homological*) *width*  $\text{Wid}_{\mathcal{H}}(M)$  to be the difference  $k_1 - k_m$  of the maximum and minimum degrees of its non-zero homology as in (2.2.1).

It is clear that the width is an invariable under shifts.

In this thesis, we only consider bounded t-structures and their hearts, and use the phrase ‘a triangulated category  $\mathcal{D}$  with heart  $\mathcal{H}$ ’ to mean that  $\mathcal{H}$  is the heart of a bounded t-structure on  $\mathcal{D}$ . Furthermore, all categories will be implicitly assumed to be  $\mathbf{k}$ -linear.

Let  $\mathcal{H}$  be a heart with corresponding t-structure  $\mathcal{P}$ . By abuse of notation, we say that an object  $P \in \mathcal{P}$  is a projective of  $\mathcal{H}$  if  $\text{Hom}_{\mathcal{D}}(P, M[1]) = 0$  for any  $M \in \mathcal{P}$ . Note that a projective of a heart is not necessary in the heart. Denote by  $\text{Proj } \mathcal{H}$  a complete set of indecomposable projectives of  $\mathcal{H}$ .

**Remark 2.2.2.** Note that the heart  $\mathcal{H}$  of a t-structure on  $\mathcal{D}$  is always an abelian category, but  $\mathcal{D}$  is not necessarily equivalent to the derived category of  $\mathcal{H}$ . On the other hand, any abelian category  $\mathcal{C}$  is the heart of a standard t-structure on  $\mathcal{D}(\mathcal{C})$ . Indeed, any object in  $\mathcal{D}(\mathcal{C})$  may be considered as a complex in  $\mathcal{C}$  and its ordinary homology objects are the factors of the filtration (2.2.1) associated to this standard t-structure. For instance,  $\mathcal{D}(Q)$  has a canonical heart  $\text{mod } \mathbf{k}Q$ , which we will write as



$\mathcal{H}_Q$  from now on. Moreover, in such cases the projectives of  $\mathcal{C}$  coincide with the normal definition.

For quivers, a convenient way to picture the categories  $\mathcal{H}_Q$  and  $\mathcal{D}(Q)$  is by drawing their Auslander-Reiten (AR) quivers.

**Definition 2.2.3.** [3, Chapter II,IV] The *AR-quiver*  $\Lambda(\mathcal{C})$  of a ( $\mathbf{k}$ -linear) category  $\mathcal{C}$  is defined as follows.

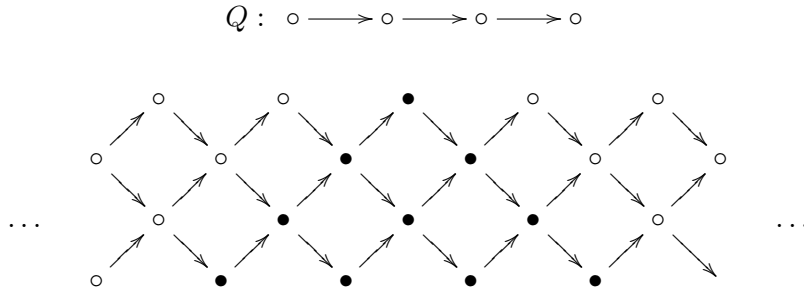
- Its vertices are identified with elements of  $\text{Ind } \mathcal{C}$ , a complete set of indecomposables of  $\mathcal{C}$ , i.e. a choice of one indecomposable object in each isomorphism class.
- The arrows from  $X$  to  $Y$  are identified with a basis of  $\text{Irr}(X, Y)$ , the space of irreducible maps  $X \rightarrow Y$  (see [3, IV 1.4 Definition]).
- There is a (maybe partially defined) bijection, called AR-translation,

$$\tau : \text{Ind } \mathcal{C} \rightarrow \text{Ind } \mathcal{C},$$

with the property that there is an arrow from  $X$  to  $Y$  if and only if there is a corresponding arrow from  $\tau Y$  to  $X$ . Moreover, we have the AR-formula

$$\text{Ext}^1(Y, X) \cong \text{Hom}(X, \tau Y)^*.$$

For example, here is (part of) the AR-quiver of  $\mathcal{D}(Q)$  for  $Q$  of type  $A_4$ . The black vertices are the AR-quiver of  $\mathcal{H}_Q$ , when  $Q$  has a straight orientation.



Besides, by the AR-formula,  $P \in \mathcal{D}(Q)$  is the projective of a heart  $\mathcal{H}$  if and only if

$$P \in \mathcal{P} \cap \tau^{-1} \mathcal{P}^\perp, \tag{2.2.3}$$

where  $\mathcal{P}$  is the t-structure corresponding to  $\mathcal{H}$ .

When  $Q$  is acyclic,  $\mathcal{H}_Q$  is hereditary, i.e.  $\text{Hom}(M, N[2]) = 0$  for any  $M, N \in \mathcal{H}_Q$ . Furthermore, one can describe the AR-quiver as a translation quiver naturally associated to  $Q$ .

**Definition 2.2.4.** ([3, Ch VIII, Def 1.1]) Let  $Q$  be a finite acyclic quiver. Define the *translation quiver*  $\mathbb{Z}Q$  as follows:

$$(\mathbb{Z}Q)_0 = \{v_j \mid v \in Q_0, j \in \mathbb{Z}\}$$

$$(\mathbb{Z}Q)_1 = \{a_j : v_j \rightarrow w_j; a'_j : w_j \rightarrow v_{j+1} \mid a : v \rightarrow w \in Q_1, j \in \mathbb{Z}\},$$

where  $Q_0, Q_1$  are the vertex and arrow set of a quiver  $Q$ . The translation  $\tau \in \text{Aut}((\mathbb{Z}Q)_0)$  is given by  $\tau(v_j) = v_{j-1}$ . Let  $\Delta$  be the underlying graph of  $Q$ . Then there is a natural projection  $\pi : \mathbb{Z}Q \rightarrow \Delta$  such that  $\pi(v_j) = \bar{v}$  and  $\pi(a_j) = \pi(a'_j) = \bar{a}$  for any  $v \in Q_0, a \in Q_1$ , where  $\bar{v}, \bar{a}$  are the corresponding vertex and edge in  $\Delta$ .

A translation quiver looks locally like:

$$\begin{array}{ccccccc} \cdots & & v_1 & & v_2 & & v_3 & & v \\ & \searrow a_0 & \nearrow a'_0 & \searrow a_1 & \nearrow a'_1 & \searrow a_2 & \nearrow a'_2 & \searrow a_3 & \downarrow \\ & w_0 & & w_1 & & w_2 & & \cdots & w \end{array} \quad (2.2.4)$$

If the underlying graph  $\Delta$  of  $Q$  is a connected tree, then  $\mathbb{Z}Q$  depends only on  $\Delta$  and will also be denoted by  $\mathbb{Z}\Delta$ . Further, a *section*  $P$  of  $\mathbb{Z}Q$  is a connected sub-quiver which meets each  $\tau$ -orbit in precisely one point. There is also a nice result due to Happel.

**Proposition 2.2.5** ([23]). *For any connected acyclic quiver  $Q$ , we have*

$$\text{Ind } \mathcal{D}(Q) = \bigcup_{m \in \mathbb{Z}} \text{Ind } \mathcal{H}_Q[m]$$

and the preprojective component of  $\Lambda(\mathcal{D}(Q))$  (i.e. the component containing the projectives of  $\mathcal{H}_Q$ ) is isomorphic to the translation quiver  $\mathbb{Z}Q$ . If  $Q$  is of Dynkin type, then we have  $\Lambda(\mathcal{D}(Q)) \cong \mathbb{Z}Q$ .

## 2.3 Tilting Theory

A similar notion to a t-structure on a triangulated category is a torsion pair on an abelian category. Tilting with respect to a torsion pair in the heart of a t-structure provides a way to pass between different t-structures.

**Definition 2.3.1.** A *torsion pair* in an abelian category  $\mathcal{C}$  is a pair of full subcategories  $\langle \mathcal{F}, \mathcal{T} \rangle$  of  $\mathcal{C}$ , such that  $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$  and furthermore every object  $E \in \mathcal{C}$  fits into a short exact sequence  $0 \rightarrow E^{\mathcal{T}} \rightarrow E \rightarrow E^{\mathcal{F}} \rightarrow 0$  for some objects  $E^{\mathcal{T}} \in \mathcal{T}$  and  $E^{\mathcal{F}} \in \mathcal{F}$ .

**Proposition 2.3.2** (Happel, Reiten, Smalø[24]). *Let  $\mathcal{H}$  be a heart in a triangulated category  $\mathcal{D}$ . Suppose further that  $\langle \mathcal{F}, \mathcal{T} \rangle$  is a torsion pair in  $\mathcal{H}$ . Then the full subcategory*

$$\mathcal{H}^\sharp = \{E \in \mathcal{D} : \mathbf{H}_1(E) \in \mathcal{F}, \mathbf{H}_0(E) \in \mathcal{T} \text{ and } \mathbf{H}_i(E) = 0 \text{ otherwise.}\}$$

*is also a heart in  $\mathcal{D}$ , as is*

$$\mathcal{H}^\flat = \{E \in \mathcal{D} : \mathbf{H}_0(E) \in \mathcal{F}, \mathbf{H}_{-1}(E) \in \mathcal{T} \text{ and } \mathbf{H}_i(E) = 0 \text{ otherwise.}\}.$$

Recall that the homology  $\mathbf{H}_\bullet$  was defined in (2.2.2). We call  $\mathcal{H}^\sharp$  the *forward tilt* of  $\mathcal{H}$ , with respect to the torsion pair  $\langle \mathcal{F}, \mathcal{T} \rangle$ , and  $\mathcal{H}^\flat$  the *backward tilt* of  $\mathcal{H}$ . Note that  $\mathcal{H}^\flat = \mathcal{H}^\sharp[-1]$ . Furthermore,  $\mathcal{H}^\sharp$  has a torsion pair  $\langle \mathcal{T}, \mathcal{F}[1] \rangle$ . With respect to this torsion pair, the forward and backward tilts are  $(\mathcal{H}^\sharp)^\flat = \mathcal{H}$  and  $(\mathcal{H}^\sharp)^\sharp = \mathcal{H}[1]$  respectively. Similarly with respect to the torsion pair  $\langle \mathcal{T}[-1], \mathcal{F} \rangle$  in  $\mathcal{H}^\flat$ , we have  $(\mathcal{H}^\flat)^\sharp = \mathcal{H}$ ,  $(\mathcal{H}^\flat)^\flat = \mathcal{H}[-1]$ .

**Proposition 2.3.3.** *Let  $M$  be an indecomposable in  $\mathcal{D}$  with canonical filtration with respect to a heart  $\mathcal{H}$ , as in (2.2.1). Given a torsion pair  $\langle \mathcal{F}, \mathcal{T} \rangle$  in  $\mathcal{H}$ , the short exact sequences*

$$0 \rightarrow H_i^\mathcal{T} \rightarrow H_i \rightarrow H_i^\mathcal{F} \rightarrow 0,$$

*can be used to refine the canonical filtration of  $M$  to a finer one with factors*

$$(H_1^\mathcal{T}[k_1], H_1^\mathcal{F}[k_1], \dots, H_m^\mathcal{T}[k_m], H_m^\mathcal{F}[k_m]), \quad (2.3.1)$$

*Furthermore, if we take the canonical filtration of  $M$  with respect to the heart  $\mathcal{H}^\sharp$  and refine it using the torsion pair  $\langle \mathcal{T}, \mathcal{F}[1] \rangle$ , then we obtain essentially the same filtration*

$$(H_1^\mathcal{T}[k_1], \tilde{H}_1^\mathcal{F}[k_1 - 1], \dots, H_m^\mathcal{T}[k_m], \tilde{H}_m^\mathcal{F}[k_m - 1]), \quad (2.3.2)$$

*where  $\tilde{H}_i = H_i[1]$ .*

*Proof.* The existence of the filtrations (2.3.1) and (2.3.2) follows by repeated use of the Octahedron Axiom.  $\square$

For two hearts  $\mathcal{H}_1$  and  $\mathcal{H}_2$  in  $\mathcal{D}$  with corresponding t-structures  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , we say  $\mathcal{H}_1 \leq \mathcal{H}_2$  if and only if  $\mathcal{P}_1 \supset \mathcal{P}_2$ , or equivalently,  $\mathcal{P}_1^\perp \subset \mathcal{P}_2^\perp$ .

**Lemma 2.3.4.** *Let  $\mathcal{H}$  be a heart in  $\mathcal{D}(Q)$ . Then  $\mathcal{H} < \mathcal{H}[m]$  for  $m > 0$ . For any forward tilt  $\mathcal{H}^\sharp$  and backward tilt  $\mathcal{H}^\flat$ , we have  $\mathcal{H}[-1] \leq \mathcal{H}^\flat \leq \mathcal{H} \leq \mathcal{H}^\sharp \leq \mathcal{H}[1]$ .*

*Proof.* Since  $\mathcal{P} \supsetneq \mathcal{P}[1]$ , we have  $\mathcal{H} < \mathcal{H}[m]$  for  $m > 0$ . By Proposition 2.3.3 we have  $\mathcal{P} \supset \mathcal{P}^\sharp$ , hence  $\mathcal{H} \leq \mathcal{H}^\sharp$ . Noticing that  $(\mathcal{H}^\sharp)^\flat = \mathcal{H}[1]$  with respect to the torsion pair  $(\mathcal{T}, \mathcal{F}[1])$ , we have  $\mathcal{H}^\sharp \leq \mathcal{H}[1]$ . Similarly,  $\mathcal{H}[-1] \leq \mathcal{H}^\flat \leq \mathcal{H}$ .  $\square$

In fact the set of all forward tilts  $\{\mathcal{H}^\sharp\}$  can be characterized as exactly all the hearts between  $\mathcal{H}$  and  $\mathcal{H}[1]$ .

**Proposition 2.3.5.** [24] *Let  $\mathcal{H}, \mathcal{H}'$  be two hearts in  $\mathcal{D}$ . Then  $\mathcal{H} \leq \mathcal{H}' \leq \mathcal{H}[1]$  if and only if  $\mathcal{H}' = \mathcal{H}^\sharp$  with respect to some torsion pair  $\langle \mathcal{F}, \mathcal{T} \rangle$  in  $\mathcal{H}$ , where  $\mathcal{T} = \mathcal{H}' \cap \mathcal{H}$  and  $\mathcal{F} = \mathcal{H}'[-1] \cap \mathcal{H}$ .*

Recall that an object in an abelian category is *simple* if it has no proper subobjects, or equivalently it is not the middle term of any (non-trivial) short exact sequence. We will denote a complete set of simples of an abelian category  $\mathcal{C}$  by  $\text{Sim } \mathcal{C}$ . Denote by  $\langle T_1, \dots, T_m \rangle$  the smallest full subcategory containing  $T_1, \dots, T_m$  and closed under extensions.

**Lemma 2.3.6.** *Let  $S$  be a simple object in an abelian Hom-finite category  $\mathcal{C}$ , with  $\text{Ext}^1(S, S) = 0$ . Then  $\mathcal{C}$  admits a torsion pair  $\langle \mathcal{F}, \mathcal{T} \rangle$  such that  $\mathcal{F} = \langle S \rangle$ . More precisely, for any  $M \in \mathcal{H}$ , in the corresponding short exact sequence*

$$0 \rightarrow M^\mathcal{T} \rightarrow M \rightarrow M^\mathcal{F} \rightarrow 0 \quad (2.3.3)$$

*we have  $M^\mathcal{F} = S \otimes \text{Hom}(M, S)^*$ . Similarly, there is also a torsion pair with the torsion part  $\mathcal{T} = \langle S \rangle$ , obtained by setting  $M^\mathcal{T} = S \otimes \text{Hom}(S, M)$ .*

*Proof.* If we define  $M^\mathcal{F}$  as in the lemma, then there is a canonical surjection  $M \rightarrow M^\mathcal{F}$ , whose kernel we may define to be  $M^\mathcal{T}$ , yielding the short exact sequence (2.3.3).

Applying  $\text{Hom}(-, S)$  to (2.3.3), we get

$$0 \rightarrow \text{Hom}(M^\mathcal{F}, S) \rightarrow \text{Hom}(M, S) \rightarrow \text{Hom}(M^\mathcal{T}, S) \rightarrow \text{Ext}^1(M^\mathcal{F}, S).$$

But

$$\begin{aligned} \text{Hom}(M^\mathcal{F}, S) &= \text{Hom}(S \otimes \text{Hom}(M, S)^*, S) \cong \text{Hom}(M, S), \\ \text{Ext}^1(M^\mathcal{F}, S) &= \text{Ext}^1(S \otimes \text{Hom}(M, S)^*, S) = 0, \end{aligned}$$

so we have  $\text{Hom}(M^\mathcal{T}, S) = 0$  and hence  $\text{Hom}(M^\mathcal{T}, M^\mathcal{F}) = 0$  as required. The proof of the second statement is similar.  $\square$

We say a forward tilting is *simple*, if the corresponding torsion free part is generated by a single simple object  $S$  with  $\text{Ext}^1(S, S) = 0$ , and denote the new heart by  $\mathcal{H}_S^\sharp$ . Similarly, a backward tilting is simple if the corresponding torsion part is generated by such a simple  $S$ , and denote the new heart by  $\mathcal{H}_S^\flat$ . In particular, for the standard heart  $\mathcal{H}_Q$  in  $\mathcal{D}(Q)$ , an APR tilting ([3, page 201]), which reverses all arrows at a sink/source of  $Q$ , is an example of a simple (forward/backward) tilting.

For later use, we fix some notations.

**Definition 2.3.7.** Let  $\mathcal{H}$  be a heart in a triangulated category  $\mathcal{D}$  with  $\mathcal{S} = \text{Sim } \mathcal{H}$ . We say  $\mathcal{H}$  is

- *finite*, if  $\mathcal{S}$  is a finite set and  $\mathcal{S}$  generates  $\mathcal{H}$  by means of extensions. More precisely, every object  $M$  in  $\mathcal{H}$  has a (finite) filtration with factors in  $\mathcal{S}$ .
- *rigid*, if any simple of  $\mathcal{H}$  is rigid.
- *spherical*, if there is an integer  $N > 1$ , such that any simple of  $\mathcal{H}$  is  $N$ -spherical, i.e.,  $\text{Hom}^\bullet(S, S) = \mathbf{k} \oplus \mathbf{k}[-N]$ .
- *monochromatic* (c.f. [11]), if for any simples  $S \neq T$  in  $\text{Sim } \mathcal{H}$ ,

$$\text{Hom}^\bullet(S, T) = \bigoplus_{j \in \mathbb{Z}} \text{Hom}^j(S, T)$$

is concentrated in a single nonnegative degree.

- *strongly monochromatic*, if it is monochromatic, and for any simples  $S \neq T$  in  $\text{Sim } \mathcal{H}$ ,  $\text{Hom}^\bullet(S, T) = 0$  or  $\text{Hom}^\bullet(T, S) = 0$ .

If a heart is finite, rigid and strongly monochromatic, we say it is *acyclic-good*; if a heart is finite, spherical and monochromatic, we say it is *Calabi-Yau good*.

For instance, it is straightforward to see that, when  $Q$  is acyclic, the standard heart  $\mathcal{H}_Q$  in  $\mathcal{D}(Q)$  is acyclic-good. Also observe that by Jordan-Hölder Theorem, the filtration in the finite condition is unique up to reorder the simple factors.

Let  $\mathcal{H}$  be an acyclic-good heart in some triangulated category with  $\text{Sim } \mathcal{H} = \{S_i\}_{i=1}^n$ . Define the *heart algebra*  $\mathcal{E}_{\mathcal{H}}$  of  $\mathcal{H}$  to be the full  $\text{Hom}^\bullet$  algebra of its simples, i.e.

$$\mathcal{E}_{\mathcal{H}} = \text{Hom}^\bullet\left(\bigoplus_{i=1}^n S_i, \bigoplus_{i=1}^n S_i\right). \quad (2.3.4)$$

We will write  $\mathcal{E}_{ij}$  for  $(\mathcal{E}_{\mathcal{H}})_{ij} = \text{Hom}^\bullet(S_i, S_j)$  if there is no ambiguity. Recall that, by [33, 10.5] and Koszul Duality, we have a derived equivalence

$$\mathcal{D}(Q) \simeq \mathcal{D}_{fd}(\text{mod } \mathcal{E}_Q), \quad (2.3.5)$$

where we consider  $\mathcal{E}_Q = \mathcal{E}_{\mathcal{H}_Q}$  as a dg algebra with trivial differential.

## 2.4 Exchange graphs

**Definition 2.4.1.** Define the *exchange graph*  $\text{EG}(\mathcal{D})$  of a triangulated category  $\mathcal{D}$  to be the oriented graph whose vertices are all hearts in  $\mathcal{D}$  and whose edges correspond to simple forward tiltings between them.

Further, let  $\mathcal{H}$  be a heart in some exchange graph  $\text{EG}(\mathcal{D})$ . Define the exchange graph  $\text{EG}_N(\mathcal{D}, \mathcal{H})$  with base  $\mathcal{H}$  to be the full subgraph of  $\text{EG}(\mathcal{D})$  induced by

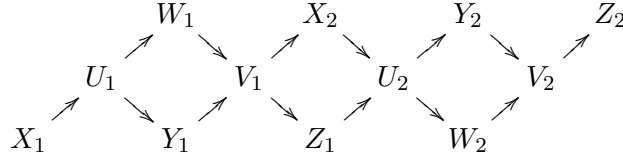
$$\{\mathcal{H}_0 \mid \mathcal{H} \in \text{EG}(\mathcal{D}), \mathcal{H}[1] \leq \mathcal{H}_0 \leq \mathcal{H}[N-1]\}.$$

We will label an edge of  $\text{EG}(\mathcal{D})$  by the simple object of the corresponding tilting, that is, the edge with end points  $\mathcal{H}$  and  $\mathcal{H}_S^\#$  will be labeled by  $S$ . Notice that  $\mathcal{H} < \mathcal{H}_S^\#$  for any simple tilting by Lemma 2.3.4, which implies there is no oriented cycle in the exchange graph.

For  $\mathcal{D}(Q)$ , denote by  $\text{EG}^\circ(Q)$  the ‘principal’ component of  $\text{EG}(\mathcal{D}(Q))$ , that is, the connected component containing  $\mathcal{H}_Q$ . Note that, if  $Q$  is of Dynkin type, we have  $\text{EG}^\circ(Q) = \text{EG}(\mathcal{D}(Q))$  (see Section 4.2), which will be denoted by  $\text{EG}(Q)$ .

In addition, denote by  $\text{EG}_N^\circ(\mathcal{D}, \mathcal{H})$ , the ‘principal’ component of  $\text{EG}_N(\mathcal{D}, \mathcal{H})$ , that is, the connected component that contains  $\mathcal{H}[1]$ . We will study  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  with base  $\mathcal{H}_Q$  and show (Corollary ??) that, in fact,  $\mathcal{H}_Q[j] \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)$  for  $1 \leq j \leq N-1$ .

**Example 2.4.2.** Let  $Q$  be the quiver of type  $A_3$  with straight orientation and  $\text{Sim } \mathcal{H}_Q = \{X, Y, Z\}$ . A piece of the AR-quiver of  $\mathcal{D}(Q)$  is as follows



where  $M_i = M[i]$  for  $M \in \text{Ind } \mathcal{H}_Q$  and  $i = 1, 2$ . Figure 2-1 is the exchange graph  $\text{EG}_3(Q, \mathcal{H}_Q)$ , where we denote a heart by its simples.

## 2.5 Cluster algebras and quiver mutation

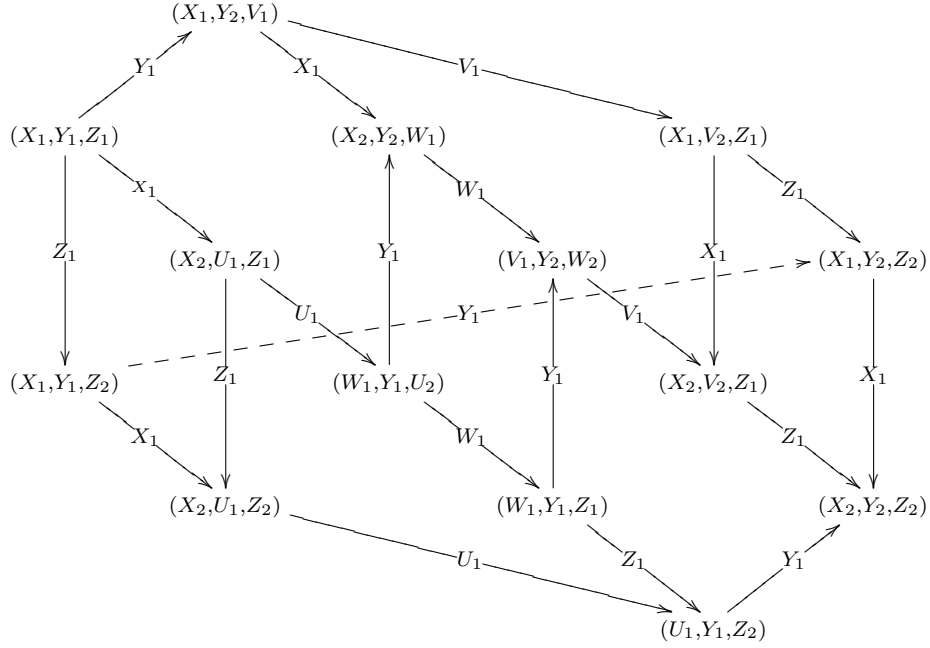
We collect some facts for cluster theory, following Fomin-Zelevinsky [16] (c.f. [29]).

**Definition 2.5.1** (Fomin-Zelevinsky). Fix an integer  $n \geq 1$ . A *seed* is a pair  $(R, \mathbf{u})$ , where

- $R$  is a finite quiver without loops or 2-cycles with vertex set  $\{1, \dots, n\}$ ;
- $\mathbf{u}$  is a free generating set  $\{u_1, \dots, u_n\}$  of the field  $\mathbb{Q}(x_1, \dots, x_n)$  of fractions of the polynomial ring  $\mathbb{Q}[x_1, \dots, x_n]$  in  $n$  indeterminates.

Let  $(R, \mathbf{u})$  be a seed and  $k$  a vertex of  $R$ . The *mutation*  $\mu_k(R, \mathbf{u})$  of  $(R, \mathbf{u})$  at  $k$  is the seed  $(R, \mathbf{u}')$ , which is obtained from  $(R, \mathbf{u})$  as follows

- $R'$  is obtained from  $R$  by



**Figure 2-1:** The exchange graph  $\text{EG}_3(Q, \mathcal{H}_Q)$  for  $Q$  of  $A_3$ -type

- 1°. adding an arrow  $i \rightarrow j$  for any pair of arrows  $(i \rightarrow k)$  and  $(k \rightarrow j)$  in  $R$ ;
- 2°. reversing all arrows incident with  $k$ ;
- 3°. deleting as many 2-cycles as possible.

- $\mathbf{u}' = (u'_1, \dots, u'_n)$  such that  $u'_i = u_i$  for  $i \neq k$  and

$$u'_k = \frac{1}{u_k} \left( \prod_{i \rightarrow k} u_i + \prod_{k \rightarrow j} u_j \right). \quad (2.5.1)$$

Notice that in the exchange relation (2.5.1), the products count the multiplicity of the arrows and they are 1 if there is no arrows coming in/out to vertex  $k$ .

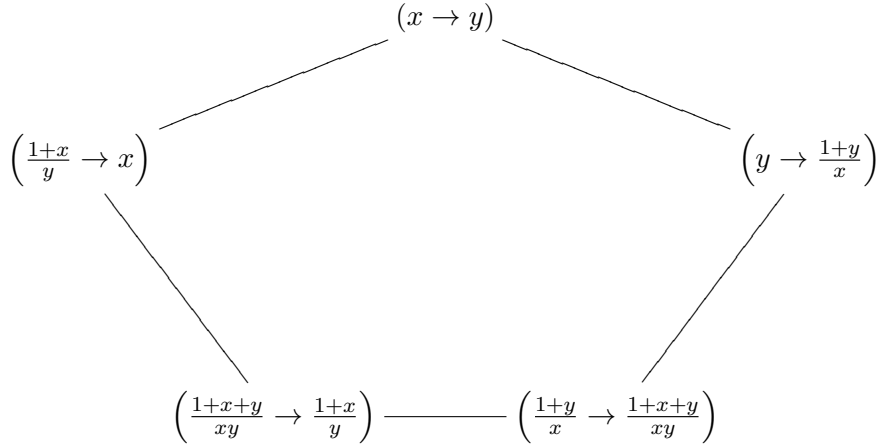
It is easy to check that the mutation is an involution, i.e.

$$\mu_k(\mu_k(R, \mathbf{u})) = (R, \mathbf{u}) \quad (2.5.2)$$

for any seed  $(R, \mathbf{u})$  and vertex  $k$ .

Let  $Q$  be a finite quiver without loops or 2-cycles with vertex set  $\{1, \dots, n\}$  within this section.

**Definition 2.5.2** (Fomin-Zelevinsky). Consider the initial seed  $(Q, \mathbf{x})$ , where  $\mathbf{x}$  consists of variables  $x_1, \dots, x_n$ . Then define



**Figure 2-2:** The cluster exchange graph of  $A_2$  type

- the *clusters* with respect to  $Q$  to be the sets  $\mathbf{u}$  appearing in seeds  $(R, \mathbf{u})$  which is iteratedly mutated from  $(Q, \mathbf{x})$ ;
- the *cluster variables* for  $Q$  to be the elements in some clusters;
- the *cluster algebra*  $\mathcal{CA}(Q)$  to be the  $\mathbb{Q}$ -subalgebra of the field  $\mathbb{Q}(x_1, \dots, x_n)$  generated by all the cluster variables.
- the (original) *cluster exchange graph*  $\text{CEG}(Q)^*$  associated with  $Q$  is the (unoriented) graph whose vertices are the seeds modulo simultaneous renumbering of the vertices and the associated cluster variables and whose edges correspond to mutations.

Note that Gekhtman-Shapiro-Vainshtein [21] showed that each cluster occurs in a unique seed in the exchange graph. By construction, if the quiver  $Q$  is mutation-equivalent to  $Q'$ , then we have an isomorphism  $\mathcal{CA}(Q) \cong \mathcal{CA}(Q')$  preserving clusters and cluster variables.

**Example 2.5.3.** Let  $R$  be  $A_2$  type quiver:  $1 \rightarrow 2$ , and consider the initial seed  $(R, \mathbf{u})$  for  $\mathbf{u} = (x, y)$ . Then Figure 2-2 is the cluster exchange graph  $\text{CEG}(R)^*$ , which is a pentagon. Moreover, the cluster exchange graph  $\text{CEG}(Q)^*$  for  $Q$  of  $A_3$ -type is the underlying graph of the exchange graph in Figure 2-1, which is a (3-dimensional) associahedron.

Fomin-Zelevinsky classified the cluster algebras of finite type as follows.

**Theorem 2.5.4** (Fomin-Zelevinsky [17]). *Let  $Q$  be a finite quiver without loops or 2-cycles with vertex set  $\{1, \dots, n\}$  and  $\mathcal{CA}(Q)$  be the associated cluster algebra.*

- All cluster variables are Laurent polynomials, i.e. their denominators are monomials.



- The number of cluster variables is finite iff  $Q$  is mutation equivalent to an orientation of a (simply laced) Dynkin diagram  $\Delta$ . In this case,  $\Delta$  is unique and the non initial cluster variables are in bijection with the positive roots of  $\Delta$ .

From Theorem 2.5.4, we know that, for Dynkin case, there is a bijection between all cluster variables and the indecomposable classes in

$$\text{Ind } \mathcal{D}(Q)/(\tau^{-1} \circ [1]).$$

This leads to the categorification of cluster algebras, which we will discuss in the next section.

## 2.6 Higher cluster categories

We recall some notations from higher cluster theory ( $m \geq 1$ ), and in particular, the relation between hearts and  $m$ -cluster tilting sets. Note that 2-cluster categories are the cluster categories corresponding to cluster algebras in Section 2.5.

**Definition 2.6.1** (c.f. [31]). Let  $m \geq 1$  be an integer and  $\Sigma_m = \tau^{-1} \circ [m-1]$  be an auto-equivalence of  $\mathcal{D}(Q)$ .

- Define the  $m$ -cluster category  $\mathcal{C}_m(Q)$  to be the orbit category  $\mathcal{D}(Q)/\Sigma_m$ . Remember that

$$\begin{aligned} \text{Ext}_{\mathcal{C}_m(Q)}^t(M, L) &= \text{Hom}_{\mathcal{C}_m(Q)}(M, L[t]) \\ &= \bigoplus_{s \in \mathbb{Z}} \text{Hom}(M, \Sigma_m^s L[t]), \end{aligned}$$

where  $\text{Hom} = \text{Hom}_{\mathcal{D}(Q)}$ . Note that (c.f. [23]), in the case when  $Q$  is acyclic, there is at most one  $t$  such that  $\text{Ext}_{\mathcal{C}_m(Q)}^t(M, L) \neq 0$ .

- An  $m$ -cluster tilting set  $\{P_i\}_{i \in I}$  in  $\mathcal{C}_m(Q)$  is an Ext-configuration, that is, a maximal collection of indecomposables such that  $\text{Ext}_{\mathcal{C}_m(Q)}^t(P_i, P_j) = 0$  for any  $1 \leq t \leq m-1$ .
- The forward mutation at the  $i$ -th object on an  $m$ -cluster tilting set  $\{P_i\}_{i=1}^n$ , consists of replacing  $P_i$  by

$$\text{Cone}(P_i \rightarrow \bigoplus_{j \neq i} \text{Irr}(P_i, P_j)^* \otimes P_j). \quad (2.6.1)$$

- Let  $\text{CEG}_m(Q)$  be the exchange graph of  $m$ -clusters, that is, the oriented graph whose vertices are  $m$ -cluster tilting sets and whose edges are the forward mutations.
- An almost complete cluster tilting object in  $\mathcal{C}_m(Q)$  is a subset of some cluster tilting set with  $n-1$  elements.

Note that, the inverse of the forward mutation at the  $i$ -th object on an  $m$ -cluster tilting set  $\{P_i\}_{i=1}^n$  is the *backward mutation*, that is, the set consists of replacing  $P_i$  by

$$\text{Cone}\left(\bigoplus_{j \neq i} \text{Irr}(P_j, P_i) \otimes P_j \rightarrow P_i\right)[-1]. \quad (2.6.2)$$

Furthermore, by the restriction of the quotient map  $\mathcal{D}(Q) \rightarrow \mathcal{C}_m(Q)$  (c.f. [53, Proposition 2.2]), we will canonically identify

$$\text{Proj } \mathcal{H}_Q[m] \cup \bigcup_{j=1}^{m-1} \text{Ind } \mathcal{H}_Q[j] \quad (2.6.3)$$

with  $\text{Ind } \mathcal{C}_m(Q)$ .

**Lemma 2.6.2.** *Let  $\mathcal{H} \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)$ . If  $\#\text{Proj } \mathcal{H} \leq \#Q_0$ , then if the equality holds, the image of  $\text{Proj } \mathcal{H}$  in  $\mathcal{C}_m(Q)$  (c.f. (2.6.3)) is identified with an  $m$ -cluster tilting set for any  $m \geq N - 1$ . In particular,  $\text{CEG}_m(Q)$  contains a ‘standard’  $m$ -cluster tilting set  $\text{Proj } \mathcal{H}_Q[1]$ .*

*Proof.* By (2.2.3),  $\text{Proj } \mathcal{H}$  is in (2.6.3) for  $m \geq N - 1$ . Then [9, Theorem 3.3] implies the lemma for  $N = 3$ , which can be modified for  $N > 3$ .  $\square$

## 2.7 Calabi-Yau categories

Let  $N > 1$  be an integer. Denote by  $\Gamma_N Q$  the *Calabi-Yau- $N$  Ginzburg (dg) algebra* associated to  $Q$ , that is, the dg algebra

$$\mathbf{k}Q_0 \langle x, x^*, e^* \mid x \in Q_1, e \in Q_0 \rangle$$

with degrees

$$\deg e = \deg x = 0, \quad \deg x^* = N - 2, \quad \deg e^* = N - 1$$

and only nontrivial differentials

$$d \sum_{e \in Q_0} e^* = \sum_{x \in Q_1} [x, x^*].$$

Write  $\mathcal{D}(\Gamma_N Q)$  for  $\mathcal{D}_{fd}(\text{mod } \Gamma_N Q)$ .

Recall that a triangulated category  $\mathcal{C}$  is called *Calabi-Yau- $N$*  if, for any objects  $L, M$  in  $\mathcal{C}$  we have a natural isomorphism

$$\mathfrak{S} : \text{Hom}_{\mathcal{C}}^\bullet(L, M) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}^\bullet(M, L)^\vee[N]. \quad (2.7.1)$$

By [37] (see also [40],[47],[50]), we know that  $\mathcal{D}(\Gamma_N Q)$  is a Calabi-Yau- $N$  category which admits a standard heart  $\mathcal{H}_\Gamma$  generated by simple  $\Gamma_N Q$ -modules  $S_e, e \in Q_0$ , each of which is  $N$ -spherical. Denote by  $\text{EG}^\circ(\Gamma_N Q)$  the principal component of the exchange graph  $\text{EG}(\mathcal{D}(\Gamma_N Q))$ , that is, the component containing  $\mathcal{H}_\Gamma$ .

## 2.8 Twist functors and braid groups

We recall (c.f. [40],[47],[50]) a distinguished family of auto-equivalences in  $\text{Aut } \mathcal{D}(\Gamma_N Q)$ , for the CY- $N$  category  $\mathcal{D}(\Gamma_N Q)$ .

**Definition 2.8.1.** The *twist functor*  $\phi$  of a spherical object  $S$  is defined by

$$\phi_S(X) = \text{Cone}(S \otimes \text{Hom}^\bullet(S, X) \rightarrow X). \quad (2.8.1)$$

with inverse

$$\phi_S^{-1}(X) = \text{Cone}(X \rightarrow S \otimes \text{Hom}^\bullet(X, S)^\vee)[-1] \quad (2.8.2)$$

The *Seidel-Thomas braid group*, denoted by  $\text{Br}(\Gamma_N Q)$ , is the subgroup of  $\text{Aut } \mathcal{D}(\Gamma_N Q)$  generating by the twist functors of the simples in  $\text{Sim } \mathcal{H}_\Gamma$ .

By [47, Lemma 2.11], if  $S_1, S_2$  are spherical, then so is  $S = \phi_{S_2}(S_1)$ . Moreover, we have

$$\phi_S = \phi_{S_2} \circ \phi_{S_1} \circ \phi_{S_2}^{-1}. \quad (2.8.3)$$

Hence the generators  $\{\phi_S \mid S \in \text{Sim } \mathcal{H}_\Gamma\}$  satisfy the braid group relations ([40],[47]) and so  $\text{Br}(\Gamma_N Q)$  is a quotient group of the braid group  $\text{Br}_Q$  associated to the quiver  $Q$ . Note that for CY-2 Dynkin or CY- $N$   $A_n$  case, we have (see [47] and [4])

$$\text{Br}_Q \cong \text{Br}(\Gamma_N Q). \quad (2.8.4)$$

## 2.9 Stability conditions

This section (following [5]) collects the basic definitions of stability conditions. Denote  $\mathcal{D}$  a triangulated category and  $K(\mathcal{D})$  its Grothendieck group.

**Definition 2.9.1** ([6] Definition 3.1). A *stability condition*  $\sigma = (Z, \mathcal{P})$  on  $\mathcal{D}$  consists of a group homomorphism  $Z : K(\mathcal{D}) \rightarrow \mathbb{C}$  called the *central charge* and full additive subcategories  $\mathcal{P}(\varphi) \subset \mathcal{D}$  for each  $\varphi \in \mathbb{R}$ , satisfying the following axioms:

- 1°. if  $0 \neq E \in \mathcal{P}(\varphi)$  then  $Z(E) = m(E) \exp(\varphi \pi i)$  for some  $m(E) \in \mathbb{R}_{>0}$ ,
- 2°. for all  $\varphi \in \mathbb{R}$ ,  $\mathcal{P}(\varphi + 1) = \mathcal{P}(\varphi)[1]$ ,

3°. if  $\varphi_1 > \varphi_2$  and  $A_i \in \mathcal{P}(\varphi_i)$  then  $\text{Hom}_{\mathcal{D}}(A_1, A_2) = 0$ ,

4°. for each nonzero object  $E \in \mathcal{D}$  there is a finite sequence of real numbers

$$\varphi_1 > \varphi_2 > \dots > \varphi_m$$

and a collection of triangles

$$0 = E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \dots \longrightarrow E_{m-1} \longrightarrow E_m = E,$$

$\swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow$   
 $A_1 \quad A_2 \quad A_m$

with  $A_j \in \mathcal{P}(\varphi_j)$  for all  $j$ .

We call the collection of subcategories  $\{\mathcal{P}(\varphi)\}$ , satisfying 2°-4° in Definition 2.9.1, the *slicing* and the collection of triangles in 4° the *Harder-Narashimhan (HN) filtration*. For any nonzero object  $E \in \mathcal{D}$  with HN-filtration above, define its upper phase to be  $\Psi_{\mathcal{P}}^+(E) = \varphi_1$  and lower phase to be  $\Psi_{\mathcal{P}}^-(E) = \varphi_n$ . By [6] Lemma 5.2,  $\mathcal{P}(\varphi)$  is abelian. An object  $E \in \mathcal{P}(\varphi)$  for some  $\varphi \in \mathbb{R}$  is said to be semistable; in which case,  $\varphi = \Psi_{\mathcal{P}}^{\pm}(E)$ . Moreover, if  $E$  is simple in  $\mathcal{P}(\varphi)$ , then it is said to be stable. Let  $I$  be an interval in  $\mathbb{R}$  and define

$$\mathcal{P}(I) = \{E \in \mathcal{D} \mid \Psi_{\mathcal{P}}^{\pm}(E) \in I\}.$$

Then for any  $\varphi \in \mathbb{R}$ ,  $\mathcal{P}([\varphi, \infty))$  and  $\mathcal{P}((\varphi, \infty))$  are t-structures in  $\mathcal{D}$ .

Denote by  $\text{Stab}(\mathcal{D})$  the set of all stability conditions on  $\mathcal{D}$ . Then there is a natural  $\mathbb{C}$  action on it, namely:

$$\Theta \cdot (Z, \mathcal{P}) = (Z \cdot z, \mathcal{P}_x),$$

where  $z = \exp(\Theta \pi \mathbf{i})$ ,  $\Theta = x + y \mathbf{i}$  and  $\mathcal{P}_x(m) = \mathcal{P}(x + m)$  for  $x, y, m \in \mathbb{R}$ . There is also a natural action on  $\text{Stab}(\mathcal{D})$  induced by  $\text{Aut}(\mathcal{D})$ , namely:

$$\xi \circ (Z, \mathcal{P}) = (Z \circ \xi, \xi \circ \mathcal{P}).$$

Similarly to stability condition on triangulated categories, we have the notation of stability function on abelian categories.

**Definition 2.9.2** ([5]). A *stability function* on an abelian category  $\mathcal{C}$  is a group homomorphism  $Z : \mathcal{K}(\mathcal{C}) \rightarrow \mathbb{C}$  such that for any object  $0 \neq M \in \mathcal{C}$ , we have  $Z(M) = m(M) \exp(\mu_Z(M) \mathbf{i} \pi)$  for some  $m(M) \in \mathbb{R}_{>0}$  and  $\mu_Z(M) \in [0, 1)$ , i.e.  $Z(M)$  lies in the upper half-plane

$$H = \{r \exp(\mathbf{i} \pi \theta) \mid r \in \mathbb{R}_{>0}, 0 \leq \theta < 1\} \subset \mathbb{C}. \quad (2.9.1)$$

Call  $\mu_Z(M)$  the phase of  $M$ . We say an object  $0 \neq M \in \mathcal{C}$  is semistable (with respect

to  $Z$ ) if every subobject  $0 \neq L$  of  $M$  satisfies  $\mu_Z(L) \leq \mu_Z(M)$ . Further, we say a stability function  $Z$  on  $\mathcal{C}$  satisfies HN-property, if for an object  $0 \neq M \in \mathcal{C}$ , there is a collection of short exact sequences

$$\begin{array}{ccccccc}
 0 = M_0 & \hookrightarrow & M_1 & \hookrightarrow & \dots & \hookrightarrow & M_{m-1} \hookrightarrow M_k = M \\
 & & \searrow & & & & \searrow \\
 & & L_1 & & & & L_k
 \end{array}$$

in  $\mathcal{C}$  such that  $L_1, \dots, L_k$  are semistable objects (with respect to  $Z$ ) and their phases are in decreasing order, i.e.  $\phi(L_1) > \dots > \phi(L_k)$ .

Note that we have a different convention  $0 \leq \theta < 1$  for the upper half plane  $H$  in (2.9.1) as Bridgeland's  $0 < \theta \leq 1$ .

Then we have another way to give a stability condition on triangulated categories.

**Proposition 2.9.3** ([5], [6]). *To give a stability condition on a triangulated category  $\mathcal{D}$  is equivalent to giving a bounded  $t$ -structure on  $\mathcal{D}$  and a stability function on its heart with the HN-property. Further, to give a stability condition on  $\mathcal{D}$  with a finite heart  $\mathcal{H}$  is equivalent to giving a function  $\text{Sim } \mathcal{H} \rightarrow H$ , where  $H$  is the upper half plane as in (2.9.1).*

Recall a crucial result of stability space.

**Theorem 2.9.4** (Bridgeland [5]). *The space  $\text{Stab}(\mathcal{D})$  of stability conditions on a triangulated category  $\mathcal{D}$  is a complex manifold.*

Therefore every finite heart  $\mathcal{H}$  corresponds to a (complex, half closed and half open)  $n$ -cell  $U(\mathcal{H}) \simeq H^n$  inside  $\text{Stab}(\mathcal{D})$ .

## CHAPTER 3

### PRELIMINARIES: ACYCLIC CASE

Throughout this chapter, let  $Q$  be an acyclic quiver with  $n$  vertices. We set up the tools for the proofs in Chapter 5

### 3.1 Change of simples

In this section, we describe how simples change during tilting, which plays a key role in the later proofs.

Define two functors as follows:

$$\psi_S^\sharp(X) = \text{Cone}(X \rightarrow S[1] \otimes \text{Ext}^1(X, S)^*)[-1], \quad (3.1.1)$$

$$\psi_S^b(X) = \text{Cone}(S[-1] \otimes \text{Ext}^1(S, X) \rightarrow X). \quad (3.1.2)$$

**Proposition 3.1.1.** *Let  $S$  be a simple in a finite heart  $\mathcal{H}$  in some triangulated category  $\mathcal{D}$  such that  $\text{Ext}^1(S, S) = 0$ . Then  $S[1]$  is a simple in  $\mathcal{H}_S^\sharp$  and  $S[-1]$  is a simple in  $\mathcal{H}_S^b$ . Moreover, for each simple  $X \not\cong S$  in  $\mathcal{H}$ , there is a simple  $\psi_S^\sharp(X)$  in  $\mathcal{H}_S^\sharp$  and a simple  $\psi_S^b(X)$  in  $\mathcal{H}_S^b$ . Further, if  $\mathcal{H}$  is finite, then so are  $\mathcal{H}_S^\sharp$  with*

$$\text{Sim } \mathcal{H}_S^\sharp = \{S[1]\} \cup \{\psi_S^\sharp(X) \mid X \in \text{Sim } \mathcal{H}, X \not\cong S\}. \quad (3.1.3)$$

and  $\mathcal{H}_S^b$  with

$$\text{Sim } \mathcal{H}_S^b = \{S[-1]\} \cup \{\psi_S^b(X) \mid X \in \text{Sim } \mathcal{H}, X \not\cong S\}. \quad (3.1.4)$$

*Proof.* We only deal with the case for forward tilting. Let  $\langle \mathcal{F}, \mathcal{T} \rangle$  be the torsion pair that corresponds to  $\mathcal{H}_S^\sharp$ . Any simple in  $\mathcal{H}_S^\sharp$  is either in  $\mathcal{T}$  or  $\mathcal{F}[1]$ . Since  $S$  has no self extension, we have  $\mathcal{F} = \{S^m \mid m \in \mathbb{N}\}$ . Furthermore, choose any simple quotient  $S_0$  of  $S[1]$  in  $\mathcal{H}_S^\sharp$ .  $S_0$  cannot be in  $\mathcal{T}$  since  $\text{Hom}(\mathcal{F}[1], \mathcal{T}) = 0$ . Thus  $S_0 \in \mathcal{F}[1]$  which implies

$S[1] = S_0$ , i.e.  $S[1] \in \text{Sim } \mathcal{H}_S^\sharp$ .

Let  $T = \psi_S^\sharp(X)$ ,  $T'$  be a simple submodule of  $X$  in  $\mathcal{H}_S^\sharp$  and  $f : T' \rightarrow X$  be a non-zero map. Since  $\text{Hom}(S[1], X) = 0$ ,  $T'$  is in  $\mathcal{T}$  instead of  $\mathcal{F}[1]$ . Because  $S$  is simple in  $\mathcal{H}$  and  $T'$  is simple in  $\mathcal{H}_S^\sharp$ , there are short exact sequences

$$0 \rightarrow L \rightarrow T' \xrightarrow{f} X \rightarrow 0 \quad (3.1.5)$$

$$0 \rightarrow T' \xrightarrow{f} X \rightarrow M \rightarrow 0$$

in  $\mathcal{H}$  and  $\mathcal{H}_S^\sharp$  respectively. Thus  $L = M[-1]$ . On the other hand  $\mathcal{H}^\sharp[-1] \cap \mathcal{H} = \mathcal{F}$ , which implies  $L \in \mathcal{F}$ . Hence we have  $L = S^m$  for some integer  $m$ .

Now applying  $\text{Hom}(-, S)$  to (3.1.5), we get

$$0 = \text{Hom}(T', S) \rightarrow \text{Hom}(L, S) \xrightarrow{g} \text{Hom}^1(X, S) \rightarrow \text{Hom}^1(T', S) = 0.$$

which implies  $g$  is an isomorphism. If  $\text{Ext}^1(X, S) = 0$ , then we have  $m = 0$  and  $T' = X = T$ . If  $\text{Ext}^1(X, S) \neq 0$ , then there is a canonical isomorphism

$$L \cong S \otimes \text{Hom}(L, S)^*.$$

hence we have a map between triangles

$$\begin{array}{ccccccc} X[-1] & \longrightarrow & S \otimes \text{Hom}(L, S)^* & \longrightarrow & T' & \longrightarrow & X \\ \parallel & & \cong \downarrow \text{id} \otimes g & & \downarrow & & \parallel \\ X[-1] & \longrightarrow & S \otimes \text{Hom}^1(X, S)^* & \longrightarrow & T & \longrightarrow & X \end{array}$$

which induces a isomorphism map  $T' \rightarrow T$ . Either way we know that  $T \cong T'$  is in  $\text{Sim } \mathcal{H}_S^\sharp$  as required.

Now suppose  $\mathcal{H}$  is finite. Then the RHS of (3.1.3) contains  $n$  non-isomorphic simples in  $\mathcal{H}_S^\sharp$ . Moreover we have  $\mathcal{K}(\mathcal{H}_S^\sharp) = \mathcal{K}(\mathcal{D}) = \text{rank } \mathcal{K}(\mathcal{H}) = n$ , where  $\mathcal{K}$  is the Grothendieck group. Thus they are all the simples in  $\mathcal{H}_S^\sharp$  as required.  $\square$

**Remark 3.1.2.** If the heart is monochromatic, then there are two cases in the equation (3.1.3) (resp. (3.1.4)):

- 1°.  $\text{Ext}^1(X, S) = 0$  (resp.  $\text{Ext}^1(S, X) = 0$ ) which implies  $X = \psi_S^\sharp(X)$  (resp.  $X = \psi_S^\flat(X)$ );
- 2°.  $\text{Hom}^\bullet(X, S) = \text{Hom}^1(X, S) \neq 0$  (resp.  $\text{Hom}^\bullet(S, X) = \text{Hom}^1(S, X) \neq 0$ ) which implies  $\psi_S^\sharp(X) = \phi_S^{-1}(X)$  (resp.  $\psi_S^\flat(X) = \phi_S(X)$ ) as in (2.8.1) and (2.8.2)

Note that the graded dual of a graded  $\mathbf{k}$ -vector space  $V = \bigoplus_{i \in \mathbb{Z}} V_i[i]$  is

$$V^\vee = \bigoplus_{i \in \mathbb{Z}} V_i^*[-i].$$

where  $V_i$  is an ungraded  $\mathbf{k}$ -vector space and  $V_i^*$  is its usual dual. Also notice (c.f. Definition 2.8.1) that, when  $S$  is spherical, (2.8.2) and (2.8.1) are the formulae for the twist functors associated to  $S$ .

**Corollary 3.1.3.** *Let  $\mathcal{H}$  be a finite and monochromatic heart with  $\text{Sim } \mathcal{H} = \{S_1, \dots, S_n\}$  in some triangulated category  $\mathcal{D}$ . For any simple  $S_i \in \text{Sim } \mathcal{H}$  such that  $\text{Ext}^1(S_i, S_i) = 0$ , define*

$$\begin{aligned} J_i^\# &= \{j \mid \text{Ext}^1(S_j, S_i) \neq 0 \mid S_j \in \text{Sim } \mathcal{H}\}, & K_i^\# &= \{1, \dots, n\} - \{i\} - J_i^\#, \\ J_i^\flat &= \{j \mid \text{Ext}^1(S_i, S_j) \neq 0 \mid S_j \in \text{Sim } \mathcal{H}\}, & K_i^\flat &= \{1, \dots, n\} - \{i\} - J_i^\flat \end{aligned}$$

and let  $T_j = \phi_{S_i}^{-1}(S_j)$ ,  $R_j = \phi_{S_i}(S_j)$ , where  $\phi^\pm$  are defined as in (2.8.2) and (2.8.1). Then we have

$$\text{Sim } \mathcal{H}_{S_i}^\# = \{S_i[1]\} \cup \{T_j\}_{j \in J_i^\#} \cup \{S_k\}_{k \in K_i^\#}, \quad (3.1.6)$$

$$\text{Sim } \mathcal{H}_{S_i}^\flat = \{S_i[-1]\} \cup \{R_j\}_{j \in J_i^\flat} \cup \{S_k\}_{k \in K_i^\flat}, \quad (3.1.7)$$

*Proof.* Follows from Proposition 3.1.1 and Remark 3.1.2.  $\square$

**Theorem 3.1.4.** *Let  $Q$  be an acyclic quiver with  $n$  vertices. Every heart in  $\text{EG}^\circ(Q)$  is acyclic-good. Moreover, any heart  $\mathcal{H}$  in  $\text{EG}^\circ(Q)$  has exactly  $n$  projectives  $\text{Proj } \mathcal{H} = \{P_1, \dots, P_n\}$ . Further, there are  $n$  simples in  $\mathcal{H}$  which can be labeled as  $S_1, \dots, S_n$  satisfying*

$$\text{Hom}^\bullet(P_i, S_j) = \delta_{ij} \mathbf{k}. \quad (3.1.8)$$

For any  $S_i \in \text{Sim } \mathcal{H}$ , we have the formula (3.1.6) and (3.1.7) as in Corollary 3.1.3 and

$$\text{Proj } \mathcal{H}_{S_i}^\# = \text{Proj } \mathcal{H} - \{P_i\} \cup \{P_i^\#\}, \quad (3.1.9)$$

$$\text{Proj } \mathcal{H}_{S_i}^\flat = \text{Proj } \mathcal{H} - \{P_i\} \cup \{P_i^\flat\}, \quad (3.1.10)$$

where  $P_i^\#$  and  $P_i^\flat$  are defined by (2.6.1) and (2.6.2) respectively.

*Proof.* Use induction, starting from the standard heart  $\mathcal{H}_Q$ , which is acyclic-good satisfying (3.1.8).

Let  $\mathcal{H}$  be an acyclic-good heart with (3.1.8) and  $S_i \in \text{Sim } \mathcal{H}$ . Corollary 3.1.3 applies to  $\mathcal{H}$  and hence  $\mathcal{H}_{S_i}^\#$  is finite with (3.1.6). By (3.1.8), for  $t \neq i$ , we have  $\text{Hom}(P_t, S_i) = 0$  which implies  $P_t$  is in  $\mathcal{P}_{S_i}^\#$  and hence in  $\text{Proj } \mathcal{H}_{S_i}^\#$ .



Next, let  $E = \text{Ext}^1(S_j, S_i)$ ,  $S_j^\sharp = \phi_{S_i}^\sharp(S_j)$ . Now, we have the following triangles

$$S_j[-1] \xrightarrow{h_j} \Omega_j \xrightarrow{\iota_j} P_j \xrightarrow{1} S_j, \quad (3.1.11)$$

$$S_j[-1] \xrightarrow{u_j} E^* \otimes S_i \xrightarrow{\theta_j} S_j^\sharp \rightarrow S_j, \quad (3.1.12)$$

$$(3.1.13)$$

where  $\Omega_j = \text{Cone}(P_j \xrightarrow{1} S_j)[-1] \in \mathcal{P}$  by last pp. Applying  $\text{Hom}(-, S_i)$  to (3.1.11) gives

$$h_j^*: \text{Hom}(\Omega_j, S_i) \xrightarrow{\cong} E, \quad (3.1.14)$$

and hence the universal map  $u_j$  lifts to the universal map

$$\alpha_j^i: \Omega_j \rightarrow \text{Hom}(\Omega_j, S_i)^* \otimes S_i \xrightarrow{\cong} E^* \otimes S_i.$$

Applying the Octahedron Axiom to the composition  $S_j[-1] \xrightarrow{h_j} \Omega_j \xrightarrow{\alpha_j} E^* \otimes S_i$  gives the following commutative diagram

$$\begin{array}{ccccccc} & & \Omega_j^i & \xlongequal{\quad} & \Omega_j^i & & \\ & & \downarrow & & \downarrow & & \\ S_j[-1] & \xrightarrow{h_j} & \Omega_j & \xrightarrow{\iota_j} & P_j & \longrightarrow & S_j \\ \parallel & & \downarrow \alpha_j^i & & \downarrow \delta & & \parallel \\ S_j[-1] & \xrightarrow{u_j} & E^* \otimes S_i & \xrightarrow{\theta_j} & S_j^\sharp & \longrightarrow & S_j \\ & & \downarrow \beta & & \downarrow & & \\ & & \Omega_j^i[1] & \xlongequal{\quad} & \Omega_j^i[1] & & \end{array} \quad (3.1.15)$$

where  $\Omega_j^i[1] = \text{Cone}(\alpha_j^i) = \text{Cone}(\delta)$ . Notice that the right square ensures  $\delta \neq 0$ . As  $S_j^\sharp$  being new simple in  $\mathcal{H}_{S_i}^\sharp$ , we deduce that  $\Omega_j^i \in \mathcal{P}_{S_i}^\sharp \subset \mathcal{P}$ .

Applying  $\text{Hom}(P_i, -)$  to (3.1.16), we get

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & (3.1.16) \\
 & & \downarrow & & \downarrow & & \\
 & & \text{Hom}(P_i, \Omega_j^i) & = & \text{Hom}(P_i, \Omega_j^i) & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \text{Hom}(P_i, \Omega_j) & \xrightarrow{\cong} & \text{Hom}(P_i, P_j) & \longrightarrow & 0 \\
 & & \downarrow (\alpha_j^i)^* & & \downarrow \delta^* & & \\
 0 & \longrightarrow & E^* & \xrightarrow{\cong} & \text{Hom}(P_i, S_j^\sharp) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Therefore, we can choose  $V_j \subset \text{Hom}(P_i, P_j)$  such that

$$V_j = \text{Irr}(P_i, P_j) = \frac{\text{Hom}(P_i, \Omega_j)}{\text{Hom}(P_i, \Omega_j^i)} \cong E^*$$

to define  $P_i^\sharp$ . A straight calculation shows that

$$\text{Hom}^\bullet(P_i^\sharp, S_j^\sharp) = 0$$

for any  $j \neq i$  and  $\text{Hom}^\bullet(P_i^\sharp, S_i[1]) = k$ , which implies  $P_i^\sharp$  is a projective in  $\text{Proj } \mathcal{H}_{S_i}^\sharp$ . By Lemma 2.6.2, these are all projectives in  $\text{Proj } \mathcal{H}_{S_i}^\sharp$ , and corresponds to some  $m$ -cluster tilting objects for  $m \gg 0$ . This implies  $P_i^\sharp$  is indecomposable and hence (3.1.9) holds. A direct calculation show that the corresponding formula (3.1.8) holds for  $\mathcal{H}_{S_i}^\sharp$ . Further, the monochromaticity of the colored quiver associated to the  $m$ -tilting objects [11, Section 2] implies the monochromaticity of  $\mathcal{H}_{S_i}^\sharp$ ; and the colored quiver is loop-free implies the rigidity of  $\mathcal{H}_{S_i}^\sharp$ . Therefore  $\mathcal{H}_{S_i}^\sharp$  is acyclic-good. Similarly (3.1.7), (3.1.10) hold and  $\mathcal{H}_{S_i}^\flat$  is acyclic-good.  $\square$

## 3.2 Convexity of exchange graphs with base

We introduce and study a linear structure of exchange graphs in this section.

For  $S \in \text{Sim } \mathcal{H}$ , inductively define

$$\mathcal{H}_S^{m\sharp} = \left( \mathcal{H}_S^{(m-1)\sharp} \right)_{S[m-1]}^\sharp$$

for  $m \geq 1$  and similarly for  $\mathcal{H}_S^{m\flat}$ ,  $m \geq 1$ . We will write  $\mathcal{H}_S^{0\sharp} = \mathcal{H}$  and  $\mathcal{H}_S^{m\sharp} = \mathcal{H}_S^{-m\flat}$  for  $m < 0$ .

**Definition 3.2.1.** A line  $l = l(\mathcal{H}, S)$  in  $\text{EG}(\mathcal{D})$ , for some triangulated category  $\mathcal{D}$ , is the full subgraph consisting of the vertices  $\{\mathcal{H}_S^{m\sharp}\}_{m \in \mathbb{Z}}$ , for some heart  $\mathcal{H}$  and a simple  $S \in \text{Sim } \mathcal{H}$ . We say an edge in  $\text{EG}(\mathcal{D})$  has *direction*  $T$  if its label is  $T[m]$  for some integer  $m$ ; we say a line  $l$  has *direction*  $T$  if some (and hence every) edge in  $l$  has direction  $T$ .

We have the following lemma.

**Lemma 3.2.2.** Let  $\mathcal{H}$  and  $\mathcal{H}_0$  be hearts in  $\text{EG}(\mathcal{D})$ . Then  $\mathcal{H}_0[1] \leq \mathcal{H} \leq \mathcal{H}_0[N-1]$  if and only if

$$\mathbf{H}_m(S) = 0, \quad \forall S \in \text{Sim } \mathcal{H}, m \notin (0, N),$$

where the homology  $\mathbf{H}_\bullet$  is with respect to  $\mathcal{H}_0$ . Moreover, if  $\mathcal{H}_0 \in \text{EG}(\mathcal{D}(Q))$  and  $\mathcal{H} \in \text{EG}_N^\circ(Q, \mathcal{H}_0)$ , then for any  $S \in \text{Sim } \mathcal{H}$ ,

$$\mathbf{H}_{N-1}(S) = 0 \quad (\text{resp. } \mathbf{H}_1(S) = 0)$$

if and only if  $\mathcal{H}_S^\sharp$  (resp.  $\mathcal{H}_S^\flat$ ) is in  $\text{EG}_N^\circ(Q, \mathcal{H}_0)$ .

*Proof.* The first assertion follows immediately from the definitions and implies immediately the sufficiency of the second assertion. By (3.1.6) in Theorem 3.1.4, we know the homology of the simples in  $\mathcal{H}_S^\sharp$  (resp.  $\mathcal{H}_S^\flat$ ) in terms of the homology of the simples in  $\mathcal{H}$ . Then the necessity of the second assertion also follows from the first one.  $\square$

An *interval* of length  $m$  in  $\text{EG}(\mathcal{D})$  is the full subgraph consisting of vertices  $\{\mathcal{H}_S^{i\flat}\}_{i=0}^{m-1}$  of some line  $l(\mathcal{H}, S)$  in  $\text{EG}(\mathcal{D})$  for some positive integer  $m$ . Notice that any interval inherits a direction from the corresponding line and in particular intervals of length one consisting of the same vertex may differ by their directions.

**Definition 3.2.3.** A *convex* subgraph  $\text{EG}_0$  of  $\text{EG}(\mathcal{D})$  is a subgraph such that for any line  $l$  in  $\text{EG}(\mathcal{D})$ , the intersection  $l \cap \text{EG}_0$  is either empty or an interval. Define the *cyclic completion* of a convex subgraph  $\text{EG}_0$  to be the oriented graph  $\overline{\text{EG}}_0$  obtained from  $\text{EG}_0$  by adding an edge  $e_l = (\mathcal{H} \rightarrow \mathcal{H}_S^{(m-1)\flat})$  with direction  $S$  for each interval  $l \cap \text{EG}_0 = \{\mathcal{H}_S^{i\flat}\}_{i=0}^{m-1}$  of direction  $S$ , in  $\text{EG}_0$ . Call the interval  $l \cap \text{EG}_0$  together with  $e_l$  a *basic cycle* (induced by  $l$  with direction  $S$ ) in  $\text{EG}_0$ .

**Proposition 3.2.4.**  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  is a convex subgraph in  $\text{EG}^\circ(Q)$ . Moreover, any basic cycle in  $\overline{\text{EG}}_N(Q, \mathcal{H}_Q)$  is an  $(N-1)$ -cycle. Further, there are a unique source  $\mathcal{H}_Q[1]$  and a unique sink  $\mathcal{H}_Q[N-1]$  in  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$ .

*Proof.* Let  $\mathcal{H} \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)$  and  $S \in \text{Sim } \mathcal{H}$ . Then  $S$  is indecomposable in  $\mathcal{D}(Q)$  and hence in  $\mathcal{H}_Q[m]$  for some integer  $1 \leq m \leq N-1$ , by Proposition 2.2.5 and the first part of Lemma 3.2.2. By the second part of Lemma 3.2.2, we have

$$l(\mathcal{H}, S) \cap \text{EG}_N^\circ(Q, \mathcal{H}_Q) = \{\mathcal{H}_S^{i\sharp}\}_{i=1-m}^{N-1-m}$$

which implies the first two statements. Let  $\mathcal{H}$  be a source in  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$ , then any simple  $S \in \text{Sim } \mathcal{H}$  satisfies  $\mathcal{H}_S^{i\sharp} \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)$ , for  $1 \leq i \leq N-2$ . By Lemma 3.2.2, we have  $\mathbf{H}_m(S) = 0$  for  $m = 2, \dots, N-1$ , where  $\mathbf{H}_\bullet$  is with respect to  $\mathcal{H}_Q$ . Thus  $\mathbf{H}_{\bullet(S)} = \mathbf{H}_1(S)$ , i.e.  $S \in \mathcal{H}_Q[1]$  and hence  $\mathcal{H} \in \mathcal{H}_Q[1]$ . Consider the corresponding t-structure, we have  $\mathcal{P} \subset \mathcal{P}_Q$ , also  $\mathcal{P}^\perp \subset \mathcal{P}_Q^\perp$ , or equivalently  $\mathcal{P} \supset \mathcal{P}_Q$ . Hence  $\mathcal{H} = \mathcal{H}_Q[1]$ , i.e.  $\mathcal{H}_Q[1]$  is the unique source. Similarly for the uniqueness of the sink.  $\square$

**Corollary 3.2.5.** *We have a canonical isomorphism*

$$\overline{\mathcal{J}} : \overline{\text{EG}}_N^\circ(Q, \mathcal{H}_Q) \cong \text{CEG}_{N-1}(Q). \quad (3.2.1)$$

between oriented graphs, which is induced by  $\mathcal{J}$  in (??). Moreover, this induces a canonical bijection between basic cycles in  $\overline{\text{EG}}_N^\circ(Q, \mathcal{H}_Q)$  and the set of almost complete cluster tilting sets.

*Proof.* For any maximal interval

$$l(\mathcal{H}, S) \cap \text{EG}_N^\circ(Q, \mathcal{H}_Q) = \{\mathcal{H}_{S_i}^{i\flat}\}_{i=0}^{N-2}$$

in  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$ , let  $\mathcal{H}_k = \mathcal{H}_{S_i}^{i\flat}$ .  $J_l = \bigcap_{i=0}^{N-2} \text{Proj } \mathcal{H}_k$ , and  $P_i^j = \text{Proj } \mathcal{H}_j - J_l$ . Note that  $S_i \in \mathcal{H}_Q[N-1]$  and

$$\text{Hom}^\bullet(P_i^j, S_i[-j]) = \text{Hom}(P_i^j, S_i[-j]) \neq 0. \quad (3.2.2)$$

By formula (3.1.10), we have  $\#J_l = n-1$  which implies  $J_l$  is an almost complete cluster tilting set. By [52, Theorem 4.3], any almost complete cluster tilting set has precisely  $N-1$  completions, and hence  $\{\text{Proj } \mathcal{H}_S^{i\flat}\}_{i=0}^{N-2}$  are all the completions of  $J_l$ .

We claim that

$$\mathcal{J}(\mathcal{H}_{j-1}) = \mu_i \mathcal{J}(\mathcal{H}_j), \quad (3.2.3)$$

for  $j = 2, \dots, N-2$ . If so, we deduce that (3.2.3) also holds for  $j = 1$  and

$$\mathcal{J}(\mathcal{H}_{N-2}) = \mu_i \mathcal{J}(\mathcal{H}_0),$$

since  $\{\mathcal{J}(\mathcal{H}_k)\}_{k=0}^{N-2}$  forms a  $(N-1)$ -cycle in  $\text{CEG}_{N-1}(Q)$  (cf. [11]). Therefore  $\mathcal{J}$  preserves edges and can be extended to the required map  $\overline{\mathcal{J}}$  that sends each new edge  $e_l = (\mathcal{H} \rightarrow \mathcal{H}_S^{(N-2)\flat})$  in any basic cycle to the mutation  $\mu_i$  on  $\mathcal{J}(\mathcal{H})$ .

To see (3.2.3), we first claim that

$$\text{Hom}_{D(Q)}(P_i^j, P) = \text{Hom}_{C_{N-1}(Q)}(P_i^j, P) \quad (3.2.4)$$

for any  $P \in J_l$ . This follows by a direct calculation, that if  $j \geq 2$ , we have  $P_i^j \in$

$\mathcal{P}^\perp[N-2]$  by (3.2.2). and hence  $\text{Hom}(\Sigma_{N-1}^t P_i^j, P) = 0$  for any  $t \neq 0$  and  $P \in J_l$ , Then we deduce that triangle

$$P_i^j \rightarrow \bigoplus_{P \in J_l} \text{Irr}(P_i^j, P)^* \otimes P \rightarrow \mu_i(P_i^j) \rightarrow P_i^j[1]$$

in  $\mathcal{C}_{N-1}(Q)$  is actually in  $\mathcal{D}(Q)$ . Hence, by applying  $\text{Hom}(-, S_i)$  to this triangle, we have  $\mu_i(P_i^j) = P_i^{j-1}$ .  $\square$

**Remark 3.2.6.** The isomorphism (3.2.1) is an interpretation of the result of Buan-Reiten-Thomas [10, Theorem 2.4].

### 3.3 Lagrangian immersions

We define a special type of functor from  $\mathcal{D}(Q)$  to  $\mathcal{D}(\Gamma_N Q)$  which induces hearts and is compatible with tilting, in the sense of Proposition 3.3.3,.

**Definition 3.3.1.** An exact functor  $F : \mathcal{D}(Q) \rightarrow \mathcal{D}(\Gamma_N Q)$  is called a *Lagrangian immersion* (L-immersion) if for any pair of objects  $(\widehat{S}, \widehat{X})$  in  $\mathcal{D}(Q)$  there is a short exact sequence

$$0 \rightarrow \text{Hom}^\bullet(\widehat{S}, \widehat{X}) \xrightarrow{F} \text{Hom}^\bullet(F(\widehat{S}), F(\widehat{X})) \xrightarrow{F^\dagger} \text{Hom}^\bullet(\widehat{X}, \widehat{S})^\vee[N] \rightarrow 0, \quad (3.3.1)$$

where  $F^\dagger = F^\vee[N] \circ \mathfrak{S}$  is the following composition

$$\text{Hom}^\bullet(F(\widehat{S}), F(\widehat{X})) \xrightarrow{\mathfrak{S}} \text{Hom}^\bullet(F(\widehat{X}), F(\widehat{S}))^\vee[N] \xrightarrow{F^\vee[N]} \text{Hom}^\bullet(\widehat{X}, \widehat{S})^\vee[N]$$

Further, we say a L-immersion is *strong* if (3.3.1) has a natural splitting.

Let  $\mathcal{H}$  be a heart in  $\mathcal{D}(\Gamma_N Q)$  with  $\text{Sim } \mathcal{H} = \{S_1, \dots, S_n\}$ . If there is a L-immersion  $F : \mathcal{D}(Q) \rightarrow \mathcal{D}(\Gamma_N Q)$  and a heart  $\widehat{\mathcal{H}} \in \text{EG}^\circ(Q)$  with  $\text{Sim } \widehat{\mathcal{H}} = \{\widehat{S}_1, \dots, \widehat{S}_n\}$ , such that  $F(\widehat{S}_i) = S_i$ , then we say that  $\mathcal{H}$  is induced via  $F$  from  $\widehat{\mathcal{H}}$  and write  $F_*(\widehat{\mathcal{H}}) = \mathcal{H}$ .

**Proposition 3.3.2.** Let  $F$  be a L-immersion and  $\widehat{\mathcal{H}}$  be an acyclic good heart in  $\text{EG}(\mathcal{D}(Q))$ . If  $\widehat{\mathcal{H}}$  induces a heart  $\mathcal{H} \in \text{EG}(\mathcal{D}(\Gamma_N Q))$  via  $F$ , i.e.  $F_*(\widehat{\mathcal{H}}) = \mathcal{H}$ , then  $\mathcal{H}$  is Calabi-Yau good. Moreover, let  $\widehat{S} \in \text{Sim } \widehat{\mathcal{H}}$  and  $S = F(\widehat{S})$ .

1°. If  $\text{Hom}^{N-1}(\widehat{S}, \widehat{X}) = 0$  for any  $\widehat{X} \in \text{Sim } \widehat{\mathcal{H}}$ , then  $F_*(\widehat{\mathcal{H}}_{\widehat{S}}^\sharp) = \mathcal{H}_S^\sharp$ .

2°. If  $\text{Hom}^{N-1}(\widehat{X}, \widehat{S}) = 0$  for any  $\widehat{X} \in \text{Sim } \widehat{\mathcal{H}}$ , then  $F_*(\widehat{\mathcal{H}}_{\widehat{S}}^\flat) = \mathcal{H}_S^\flat$ .

*Proof.* First, the finiteness of  $\widehat{\mathcal{H}}$  implies the finiteness of  $\mathcal{H}$ . Second, by (3.3.1), the strongly monochromaticity and rigidity of  $\widehat{\mathcal{H}}$  implies the monochromaticity and sphericity of  $\mathcal{H}$ . Thus  $\mathcal{H}$  is Calabi-Yau good.

For any  $\widehat{X}(\not\cong \widehat{S})$  in  $\text{Sim } \widehat{\mathcal{H}}$ , let  $X = F(\widehat{X})$ . Since  $\text{Hom}^{N-1}(\widehat{S}, \widehat{X}) = 0$ , the short exact sequence (3.3.1) becomes an isomorphism  $F : \text{Hom}^\bullet(\widehat{S}, \widehat{X}) \xrightarrow{\sim} \text{Hom}^\bullet(S, X)$ . Since  $F$  is exact, we have

$$F(\psi_{\widehat{S}}^\#(\widehat{X})) = \psi_S^\#(X),$$

where  $\psi^\#$  is defined as in (3.1.1). Then by Proposition 3.1.1 we have  $F_*(\widehat{\mathcal{H}}_{\widehat{S}}^\#) = \mathcal{H}_S^\#$ . Similarly for  $2^\circ$ .  $\square$

When there is no ambiguity, we will write good for either acyclic-good or Calabi-Yau good.

## CHAPTER 4

### PRELIMINARIES: DYNKIN CASE

Throughout this chapter, let  $Q$  be a Dynkin quiver with  $n$  vertices. We will set up the tools for Chapter 6. As part of this setting, we will reproduce the proof of the connectedness of the exchange graph  $\mathcal{D}(Q)$ .

#### 4.1 Standard hearts in $\mathbf{D}(Q)$

We will give several characterization of standard hearts in  $\mathcal{D}(Q)$  in this section.

Following [3, Chapter IX], we introduce several notions:

- A *path* in  $\Lambda(\mathcal{C})$  is a sequence

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \rightarrow \dots \xrightarrow{f_{t-1}} M_{t-1} \xrightarrow{f_t} M_t$$

of irreducible maps  $f_i$  between indecomposable modules  $M_i$  with  $t \geq 1$ . When such a path exists, we say that  $M_0$  is a *predecessor* of  $M_t$  or  $M_t$  is a *successor* of  $M_0$ .

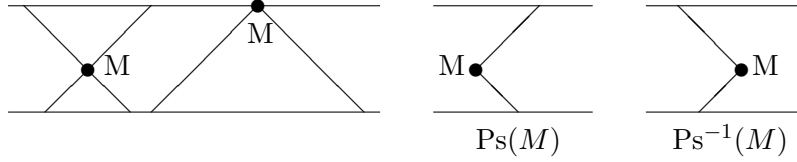
- A *cycle* in  $\Lambda(\mathcal{C})$  is a path with  $M_0 = M_t$ . An indecomposable in  $\mathcal{C}$  is called *directing* if it is not in any cycle.
- A path  $M_0 \rightarrow \dots \rightarrow M_t$  in  $\Lambda(\mathcal{C})$  is called *sectional* if, for all  $1 < i \leq t$ ,  $\tau M_i \not\cong M_{i-2}$ .

In  $\mathbb{Z}Q$ , define

$$\text{Ps}(M) = \{\text{objects in some sectional path starting from } M\},$$

$$\text{Ps}^{-1}(M) = \{\text{objects in some sectional path ending at } M\}.$$

For example, the sectional paths in a  $A$ -type AR-quiver in the derived category looks like:



Now, we have the following lemma.

**Lemma 4.1.1.** *We have*

- 1°. *Any section in  $\mathbb{Z}Q$  is isomorphic to some orientation of  $\Delta$ .*
- 2°. *For any object  $M$  in  $\mathbb{Z}Q$ ,  $\text{Ps}(M)$  and  $\text{Ps}^{-1}(M)$  are sections.*
- 3°. *The projectives of  $\mathcal{H}_Q$  together with the irreducible maps between them are a section in  $\Lambda(\mathcal{D}(Q))$ . Moreover the section has the exactly the opposite orientation of  $Q$ .*

*Proof.* The first two assertions follow directly from the definition of section and sectional path. For the last one, c.f.[3].  $\square$

For a section  $P$  in  $\Lambda(\mathcal{D}(Q)) \cong \mathbb{Z}Q$ , define

$$[P, \infty) = \bigcup_{m \geq 0} \tau^{-m} P = \{M \mid \tau^m(M) \in P, \text{ for some } m \geq 0\}$$

Similarly for  $(-\infty, P]$ . Also define  $[P_1, P_2] = [P_1, \infty) \cap (-\infty, P_2]$ .

The following lemma characterizes one such type of interval.

**Lemma 4.1.2.** *The interval  $[\text{Ps}(M), \infty)$  consists precisely all the successors of  $M$ . Similarly,  $(-\infty, \text{Ps}^{-1}(M)]$  consists precisely all the predecessors of  $M$ .*

*Proof.* We only prove the first assertion. The second is similar.

By the local property (2.2.4) of the translation quiver  $\mathbb{Z}Q$ , any object in  $[\text{Ps}(M), \infty)$  is a successor of  $M$ . On the other hand, let  $L$  be any successor of  $M$  with path

$$M = M_0 \xrightarrow{f_1} M_1 \rightarrow \dots \xrightarrow{f_j} M_j = L.$$

If  $\tau M_i = M_{i-2}$  for some  $3 \leq i \leq j$ , then consider  $\tau L$  with path

$$M = M_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-2}} M_{i-2} = \tau M_i \xrightarrow{\tau f_i} \tau M_{i+1} \xrightarrow{\tau f_{i+1}} \dots \xrightarrow{\tau f_j} \tau M_j = \tau L.$$

we can repeat the process until the path is sectional, i.e. until we obtain  $\tau^k L \in \text{Ps}(M)$  for some  $k \geq 0$ . Thus  $L \in [\text{Ps}(M), \infty)$ .  $\square$



**Lemma 4.1.3.** *Let  $M, L \in \text{Ind } \mathcal{D}(Q)$ . If  $\text{Hom}(M, L) \neq 0$  then*

$$\begin{aligned} L &\in [\text{Ps}(M), \text{Ps}^{-1}(\tau(M[1]))], \\ M &\in [\text{Ps}(\tau^{-1}(L[-1])), \text{Ps}^{-1}(L)]. \end{aligned}$$

*Proof.* By the Auslander-Reiten formula, we have

$$\text{Hom}(M, L)^* = \text{Hom}(\tau^{-1}(L), M[1]).$$

The lemma now follows from Lemma 4.1.2.  $\square$

**Proposition 4.1.4.** *A section  $P$  in  $\mathcal{D}(Q)$  will induce a unique t-structure  $\mathcal{P}$  on  $\mathcal{D}(Q)$  such that  $\text{Ind } \mathcal{P} = [P, \infty)$ . For any t-structure  $\mathcal{P}$  on  $\mathcal{D}(Q)$ , the followings are equivalent*

1°.  $\mathcal{P}$  is induced by some section  $P$ .

2°.  $\text{Ind } \mathcal{D}(Q) = \text{Ind } \mathcal{P} \cup \text{Ind } \mathcal{P}^\perp$ .

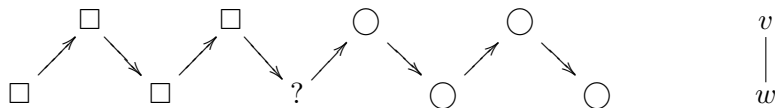
3°. The corresponding heart  $\mathcal{H}$  is isomorphic to  $\mathcal{H}_{Q'}$ , where  $Q'$  has the same underlying diagram of  $Q$ .

4°.  $\text{Wid}_{\mathcal{H}} M = 0$  for any  $M \in \text{Ind } \mathcal{D}(Q)$ , where  $\mathcal{H}$  is the corresponding heart.

*Proof.* For a section  $P$ , let  $\mathcal{P}$  be the subcategory which is generated by the elements in  $\text{Ind } \mathcal{P} = [P, \infty)$ . Notice that  $\text{Ind } \mathcal{P}^\perp = (\infty, \tau^{-1}P]$  which implies  $\mathcal{P}$  is a t-structure. Thus  $1^\circ \Rightarrow 2^\circ$ . Since  $\mathcal{H} = [P, P[1])$ ,  $1^\circ \Rightarrow 3^\circ$ .

If  $\mathcal{H}$  is isomorphic to  $\mathcal{H}'_{Q'}$  for some quiver  $Q'$ , then  $\text{Ind } \mathcal{P} = \cup_{j \geq 0} \mathcal{H}[j] = [P', \infty)$ , where  $P'$  is the sub-quiver in  $\Lambda(\mathcal{D}(Q))$  consists of the projectives. Thus  $3^\circ \Rightarrow 1^\circ$ . Since for any  $M \in \text{Ind } \mathcal{D}(Q)$ ,  $\text{Wid}_{\mathcal{H}} M = 0$  if and only if  $M \in \mathcal{H}[k]$  for some integer  $k$ , we have  $3^\circ \Rightarrow 4^\circ$ . Notice that  $\mathcal{H}[k]$  is either in  $\mathcal{P}$  or  $\mathcal{P}^\perp$ , we have  $4^\circ \Rightarrow 2^\circ$ .

Now we only need to prove  $2^\circ \Rightarrow 1^\circ$ . If an indecomposable  $M$  is in  $\mathcal{P}$  (resp.  $\mathcal{P}^\perp$ ), then, inductively, all of its successors (resp. predecessors) are in  $\mathcal{P}$  (resp.  $\mathcal{P}^\perp$ ). By the local property (2.2.4),  $\tau^m(M)$  is a successor of  $M$  if  $m \geq 0$  and a predecessor if  $m \leq 0$ . Hence, in any row  $\pi^{-1}(v) \in \mathbb{Z}Q \cong \Lambda(\mathcal{D}(Q))$ , for any vertex  $v \in Q_0$ , there is a unique integer  $m_v$  such that  $\tau^j(v) \in \mathcal{P}$ , for  $j \geq m_v$ , while  $\tau^j(v) \in \mathcal{P}^\perp$ , for  $j < m_v$ . Furthermore, for a neighboring vertex  $w$  of  $v$ , the local picture looks like this



where  $\bigcirc \in \mathcal{P}$  and  $\square \in \mathcal{P}^\perp$ . Hence  $v_{m_v}$  and  $w_{m_w}$  must be connected in  $\mathbb{Z}Q$  and so the full sub-quiver of  $\mathbb{Z}Q$  consisting of all vertices  $\{v_{m_v}\}_{v \in Q_0}$  is a section and furthermore it induces  $\mathcal{P}$ .  $\square$

We call a heart on  $\mathcal{D}(Q)$  is *standard* if the corresponding t-structure is induced by a section.

For later use, we define the position function as follows.

**Definition/Lemma 4.1.5.** There is a *position function*  $\text{pf} : \Lambda(\mathcal{D}(Q)) \rightarrow \mathbb{Z}$ , unique up to an additive constant, such that  $\text{pf}(M) - \text{pf}(\tau M) = 2$  for any  $M \in \Lambda(\mathcal{D}(Q))$  and  $\text{pf}(M) < \text{pf}(L)$  for any successor  $L$  of  $M$ . For a heart  $\mathcal{H}$  in  $\text{EG}(Q)$ , define

$$\text{pf}(\mathcal{H}) = \sum_{S \in \text{Sim } \mathcal{H}} \text{pf}(S).$$

## 4.2 Connectness of $\text{EG}(Q)$

We give two proofs of the connectedness of the exchange graph for  $\mathcal{D}(Q)$ , which was a result of Keller-Vossieck [38].

We say an indecomposable object  $L$  in a subcategory  $\mathcal{C} \subset \mathcal{D}(Q)$  is *leftmost* if there is no path from any other indecomposable in  $\mathcal{C}$  to  $L$ , or equivalently that no predecessor of  $L$  is in  $\mathcal{C}$ . In particular, a leftmost object in a heart is simple. If in a simple forward tilting, the simple object is leftmost, we call it a *L-tilting*. Similarly, an indecomposable object  $R$  is *rightmost* if there is no path from any other indecomposable to  $L$ .

**Lemma 4.2.1.** *Let  $S$  be leftmost in  $\mathcal{H}$  and  $\mathcal{H}^\sharp = \mathcal{H}_S^\sharp$ . We have*

$$1^\circ. (\text{Ind } \mathcal{H} \setminus \{S\}) \subset \mathcal{H}^\sharp.$$

$$2^\circ. \text{ Following the notation in Proposition 2.3.3. If } m > 1, \text{ then } H_m^\mathcal{F} = 0.$$

$$3^\circ. \text{ For any } M \in \text{Ind } \mathcal{D}(Q), \text{ Wid}_{\mathcal{H}^\sharp} M \leq \text{Wid}_{\mathcal{H}} M.$$

*Proof.* Since  $S$  is a leftmost object, then  $\text{Ind } \mathcal{F} = \{S\}$  and  $\mathcal{F} = \{S^i \mid i \in \mathbb{Z}^+\}$ . For any indecomposable in  $\mathcal{H}$  other than  $S$ , we have  $\text{Hom}(M, S) = 0$  which implies  $(\text{Ind } \mathcal{H} \setminus \{S\}) \subset \mathcal{T} \subset \mathcal{H}^\sharp$ .

For  $2^\circ$ , suppose  $H_m^\mathcal{F} = S^j \neq 0$ , then  $M[-k_m]$  is the predecessor of  $S$ . Consider an indecomposable summand  $L$  of  $H_1$ . If  $L = S$ , then  $S[k_1]$  is the predecessor of  $M$ . Since  $k_1 > k_m$ ,  $S$  is the predecessor of  $S[k_1 - k_m]$ , hence the predecessor of  $M[-k_m]$ . Then  $M$  and  $S$  are predecessors to each other which is a contradiction. If  $L \neq S$ , then  $L \in \mathcal{T}$ .  $L$  is the predecessor of  $M[-k_1]$ , hence the predecessor of  $M[-k_m]$ . Then  $L$  is the predecessor of  $S$  which is also a contradiction.

For  $3^\circ$ , if  $\text{Wid}_{\mathcal{H}} M > 0$ , then  $m > 1$ . By  $2^\circ$ ,  $H_m^\mathcal{F} = 0$ . Then by (2.3.2),  $\text{Wid}_{\mathcal{H}^\sharp} M \leq k_1 - k_m = \text{Wid}_{\mathcal{H}} M$ . If  $\text{Wid}_{\mathcal{H}} M = 0$ , or equivalently  $m = 1$ , then by (2.3.2) again,  $\text{Wid}_{\mathcal{H}^\sharp} M = 0 = \text{Wid}_{\mathcal{H}} M$ .  $\square$

By the same argument in the proof of Lemma 4.2.1  $2^\circ$ , we know that an object  $S$  is a leftmost object in a heart  $\mathcal{H}$  in  $\mathcal{D}(Q)$ , if and only if it is a leftmost object in the corresponding t-structure  $\mathcal{P}$ .

**Corollary 4.2.2.** *For a L-tilting with respect to a leftmost object  $S$ , we have  $\text{Ind } \mathcal{P}^\sharp = \text{Ind } \mathcal{P} - \{S\}$ .*

*Proof.* Consider the filtration 2.3.1, we have  $M \notin \mathcal{P}$  if and only if  $k_m < 0$ . If so, since  $H_m^\mathcal{T}$  or  $H_m^\mathcal{F}$  is not 0 in the filtration 2.3.1, then  $M \notin \mathcal{P}^\sharp$ . Thus  $\text{Ind } \mathcal{P}^\sharp \subset \text{Ind } \mathcal{P}$ . On the other hand,  $M \in \text{Ind } \mathcal{P} - \text{Ind } \mathcal{P}^\sharp$  if and only if  $H_m^\mathcal{F} \neq 0$  and  $k_m = 0$ . In which case,  $m = 1$  by Lemma 4.2.1, and hence  $M = S$ .  $\square$

**Lemma 4.2.3.** *For any object  $M \in \text{Ind } \mathcal{D}(Q)$ , if  $\text{Wid}_\mathcal{H} M > 0$ , then applying any sequence of L-tiltings to  $\mathcal{H}$  must reduce the width of  $M$  after finitely many steps.*

*Proof.* Suppose not, let  $\text{Wid}_\mathcal{H} M > 0$  is the minimal width of  $M$  under any L-tilting. We have  $m > 1$  in filtration (2.3.1). Then  $H_m^\mathcal{F} = 0$  by Lemma 4.2.1. In the filtration (2.3.2), if  $H_1^\mathcal{T}$  vanishes, then  $\text{Wid}_{\mathcal{H}^\sharp} M \leq (k_1 - 1) - k_m < \text{Wid}_\mathcal{H} M$ . But  $\text{Wid}_\mathcal{H} M$  is minimal, thus  $H_1^\mathcal{T} \neq 0$  after any L-tilting.

Consider the size of  $H_1^\mathcal{T}$ . Let  $H_1^\mathcal{T} = \oplus_{j=1}^l T_j^{s_j}$ , where  $T_j$  are different indecomposables in  $\mathcal{T}$  and  $l$  is a positive integer. By (2.3.2) we know  $H_1^\mathcal{T}$  will not change if we do L-tilting that is not with respect to any  $T_j$ . And if we do L-tilting with respect to some  $T_j$ , then  $H_1^\mathcal{T}$  will lose the summand  $T_j$ . Since  $H_1^\mathcal{T}$  can not vanish, we can assume after many L-tilting,  $l$  reaches the minimum, and we can not do L-tilting that is with respect to any  $T_i$ .

On the other hand, for any object  $M \in \text{Ind } \mathcal{D}(Q)$ , while  $M[m]$  is the successor of some  $T_j$  when  $m$  is large enough, it can not be leftmost in any heart that contains  $T_j$ . Besides we can only do L-tilting with respect to any object once. Thus, we will eventually have to tilt  $T_i$ , which will reduce  $l$  and it is a contradiction.  $\square$

Now we have a proposition about how one can do L-tilting.

**Proposition 4.2.4.** *Applying any sequence of L-tiltings to any heart, will make it standard after finitely many steps.*

*Proof.* By Lemma 4.2.3, the width of any particular indecomposable will become zero after finitely steps in the sequence. But, up to shift, there are only finitely many indecomposables in  $\text{Ind } \mathcal{D}(Q)$ . Thus, after finitely steps, we must reach a heart with respect to which all indecomposables have width zero and which is therefore standard, by Proposition 4.1.4.  $\square$

Now we can prove the connectedness:

**Theorem 4.2.5** ([38]).  *$\text{EG}(Q)$  is connected.*

*Proof.* Since t-structure and heart are one-one correspondent, any heart is connected to a standard heart by Proposition 4.2.4. On the other hand, using the equivalent definition 3° in Proposition 4.1.4 for ‘standard’, the set of all standard hearts is connected by APR-tilting (c.f.[3, page 201]). So the theorem follows.  $\square$

### 4.3 Mutation rules for heart algebras

**Proposition 4.3.1.** *Let  $\mathcal{H}$  be a heart of  $\mathcal{D}(Q)$  with  $\text{Sim } \mathcal{H} = \{S_1, \dots, S_n\}$  and  $\mathcal{E}_{ij} = \text{Hom}^\bullet(S_i, S_j)$ . Then for  $i \neq j, j \neq k$ ,*

$$1^\circ. \dim \mathcal{E}_{ij} + \dim \mathcal{E}_{ji} \leq 1.$$

$$2^\circ. \text{ If } \mathcal{E}_{ij}, \mathcal{E}_{jk}, \mathcal{E}_{ik} \neq 0, \text{ then the multiplication } \mathcal{E}_{ij} \otimes \mathcal{E}_{jk} \rightarrow \mathcal{E}_{ik} \text{ is an isomorphism.}$$

*Proof.* Suppose that  $\mathcal{E}_{ij}^{\delta_1} \neq 0$  for some  $\delta_1 > 0$ . Let  $A = S_i$  and  $B = S_j[\delta_1]$ . By Lemma 4.1.3, we have

$$B \in \left[ \text{Ps}(A), \text{Ps}^{-1}(\tau(A[1])) \right].$$

Thus  $\mathcal{E}_{ij}^m = 0$  for  $m \neq \delta_1$  and  $\mathcal{E}_{ji}^m = 0$  for  $m > 1 - \delta_1$ . But  $\mathcal{E}_{ji}$  is also concentrated in positive degrees and hence  $\mathcal{E}_{ji} = 0$ .

By Proposition 4.1.4, there is a quiver  $Q'$  such that,  $\text{Ps}(A)$  consists precisely of the projectives in  $\text{mod } \mathbf{k}Q'$ . Moreover, we have  $B \in \text{mod } \mathbf{k}Q'$ . Let  $\mathbf{b} = \underline{\dim} B$  and  $\mathbf{a} = \underline{\dim} A$ , then we have

$$\begin{cases} \dim \text{Hom}(A, B) - \dim \text{Ext}^1(A, B) = \langle \mathbf{a}, \mathbf{b} \rangle = \dim \mathcal{E}_{ij}^{\delta_1}, \\ \dim \text{Hom}(B, A) - \dim \text{Ext}^1(B, A) = \langle \mathbf{b}, \mathbf{a} \rangle = \dim \mathcal{E}_{ji}^{\delta_1} = 0. \end{cases} \quad (4.3.1)$$

Since  $Q'$  is of Dynkin type, the quadratic form  $q(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle$  is positive definite and, furthermore, since  $A \not\cong B$ , we have  $\mathbf{a} \neq \mathbf{b}$ . Hence

$$0 < \langle \mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{b} \rangle = 2 - \langle \mathbf{a}, \mathbf{b} \rangle$$

i.e.  $\dim \mathcal{E}_{ij}^{\delta_1} \leq 1$ . Thus  $1^\circ$  follows.

For  $2^\circ$ , suppose that  $\mathcal{E}_{jk}^{\delta_2} \neq 0$ . Since  $B \in \mathcal{H}'_Q$ , Lemma 4.1.3 implies that

$$S_k[\delta_1 + \delta_2] \in (\mathcal{H}'_Q)[1] \cup \mathcal{H}'_Q.$$

Suppose that  $\mathcal{E}_{ik}^{\delta_3} \neq 0$  and we have  $C = S_k[\delta_3]$  is also in  $\mathcal{H}'_Q$ . Thus either  $\delta_3 = \delta_1 + \delta_2$  or  $\delta_3 = \delta_1 + \delta_2 - 1$ .

Suppose that  $\delta_3 = \delta_1 + \delta_2 - 1$ . Let  $\mathbf{c} = \underline{\dim} C$ . As in (4.3.1), we have

$$\begin{cases} \langle \mathbf{a}, \mathbf{b} \rangle = 1, & \begin{cases} \langle \mathbf{a}, \mathbf{c} \rangle = 1, \\ \langle \mathbf{b}, \mathbf{a} \rangle = 0, \end{cases} & \begin{cases} \langle \mathbf{b}, \mathbf{c} \rangle = -1, \\ \langle \mathbf{c}, \mathbf{a} \rangle = 0, \end{cases} \end{cases}$$

Because  $A$  is simple,  $\mathbf{a} \neq \mathbf{b} + \mathbf{c}$ . But  $\langle \mathbf{b} + \mathbf{c} - \mathbf{a}, \mathbf{b} + \mathbf{c} - \mathbf{a} \rangle = 0$ , which is a contradiction. Therefore  $\delta_3 = \delta_1 + \delta_2$ .

Since  $A$  is a simple, any non-zero  $f \in \text{Hom}(A, B)$  is injective and so gives a short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow D \rightarrow 0$  in  $\text{mod } \mathbf{k}Q'$ . Applying  $\text{Hom}(-, C)$  to it, we get

an exact sequence

$$\begin{aligned} 0 \rightarrow \operatorname{Hom}(D, C) \rightarrow \operatorname{Hom}(B, C) \xrightarrow{f^*} \operatorname{Hom}(A, C) \rightarrow \\ \rightarrow \operatorname{Hom}(D, C[1]) \rightarrow \operatorname{Hom}(B, C[1]) = 0 \end{aligned}$$

If  $f^*$  is not an isomorphism, then  $\operatorname{Hom}(D, C) \neq 0$  and  $\operatorname{Hom}(D, C[1]) \neq 0$ , contradicting Lemma 4.1.3. Hence multiplication  $\mathcal{E}_{ij} \otimes \mathcal{E}_{jk} \rightarrow \mathcal{E}_{ik}$ , i.e. composition  $\operatorname{Hom}(A, B) \otimes \operatorname{Hom}(B, C) \rightarrow \operatorname{Hom}(A, C)$ , is an isomorphism, as required.  $\square$

Next we describe the heart algebra after mutation. When there is an unique (non-trivial) extension of  $S_i$  on top of  $S_j$ , we will use  $S_i \triangle S_j$  to denote the corresponding object. Recall that

$$\dim(\mathcal{E}_{ab} \oplus \mathcal{E}_{ba}) \leq 1.$$

**Theorem 4.3.2.** *Let  $\mathcal{H}$  be a heart in  $\operatorname{EG}(Q)$ ,  $S_i \in \operatorname{Sim} \mathcal{H}$ . Denote by  $\mathcal{E}$  and  $\mathcal{E}^\#$  the heart algebra of  $\mathcal{H}$  and  $\mathcal{H}_{S_i}^\#$  respectively. Define*

$$J_i^\# = \{j \mid \mathcal{E}_{ji}^1 \neq 0\}, \quad K_i^\# = \{j \mid \mathcal{E}_{ji}^1 = 0, j \neq i\}.$$

For  $j \in J_i^\#, k \in K_i^\#, \forall \delta \in \mathbb{Z}$ , We have

$$I^\circ. (\mathcal{E}^\#)_{ij}^1 = (\mathcal{E}_{ji}^1)^*, \quad (\mathcal{E}^\#)_{ik} = \mathcal{E}_{ik}[-1], \quad (\mathcal{E}^\#)_{ki} = \mathcal{E}_{ki}[1].$$

$$\mathcal{P}^\circ. \text{ If } \mathcal{E}_{ki} \oplus \mathcal{E}_{ik} \neq 0 \text{ and } \mathcal{E}_{kj} \oplus \mathcal{E}_{jk} \neq 0, \text{ then } (\mathcal{E}^\#)_{kj} \oplus (\mathcal{E}^\#)_{jk} = 0.$$

$$\mathcal{P}^\circ. \text{ If } \mathcal{E}_{ki} \oplus \mathcal{E}_{ik} \neq 0 \text{ and } \mathcal{E}_{kj} \oplus \mathcal{E}_{jk} = 0, \text{ then}$$

$$(\mathcal{E}^\#)_{kj}^\delta = \mathcal{E}_{ki}^\delta \otimes (\mathcal{E}_{ji}^1)^*, \quad (\mathcal{E}^\#)_{jk}^\delta = \mathcal{E}_{ji}^1 \otimes \mathcal{E}_{ik}^\delta.$$

For all other cases,  $(\mathcal{E}^\#)_{lm} = \mathcal{E}_{lm}$ .

*Proof.* Let  $\operatorname{Sim} \mathcal{H} = \{S_1, \dots, S_n\}$ . By Theorem 3.1.4 We have

$$\operatorname{Sim} \mathcal{H}_{S_i}^\# = \{S_i[1]\} \cup \{T_j\}_{j \in J_i^\#} \cup \{S_k\}_{k \in K_i^\#},$$

where

$$T_j = \operatorname{Cone}(S_j \rightarrow S_i \otimes \operatorname{Hom}^\bullet(S_j, S_i)^\vee)[-1].$$

In our case we have  $T_j = (S_i \otimes V_j) \triangle S_j$  for  $V_j = \operatorname{Ext}^1(S_i, S_j) \cong \mathbf{k}$ .

First,  $\text{Hom}^\bullet(S_i, S_j) = \mathcal{E}_{ij} = 0$  by Proposition 4.3.1 and we have

$$\begin{aligned} (\mathcal{E}^\#)_{ij}^1 &= \text{Hom}^1(S_i[1], T_j) \\ &= \text{Hom}^1(S_i[1], (S_i \otimes V_j) \triangle S_j) \\ &= \text{Hom}(S_i[1], S_i[1] \otimes V_j) \\ &= V_j = (\mathcal{E}_{ji}^1)^*. \end{aligned}$$

The rest of 1° and  $(\mathcal{E}^\#)_{lm} = \mathcal{E}_{lm}$  for  $l, m \in (K_i^\# \cup \{i\})$ , follows directly. For  $l, m \in J_i^\#$ , by Proposition 4.3.1 we have

$$\text{Hom}^\bullet(S_l, S_i) = 0 \quad \text{Hom}^\bullet(S_i, S_m) = 0.$$

Thus

$$\begin{aligned} \text{Hom}^\bullet(T_l, T_m) &= \text{Hom}^\bullet((S_i \otimes V_l) \triangle S_l, (S_i \otimes V_m) \triangle S_m) \\ &= \text{Hom}^\bullet((S_i \otimes V_l) \triangle S_l, S_m) \\ &= \text{Hom}^\bullet(S_l, S_m), \end{aligned}$$

i.e.  $(\mathcal{E}^\#)_{lm} = \mathcal{E}_{lm}$ . Now let  $j \in J_i^\#, k \in K_i^\#$ . If  $\mathcal{E}_{ik} = \mathcal{E}_{ki} = 0$ , then

$$\text{Hom}^\bullet(T_j, S_k) = \text{Hom}^\bullet((S_i \otimes V_j) \triangle S_j, S_k) = \text{Hom}^\bullet(S_j, S_k)$$

i.e.  $(\mathcal{E}^\#)_{jk} = \mathcal{E}_{jk}$ . Similarly  $(\mathcal{E}^\#)_{kj} = \mathcal{E}_{kj}$ .

Now suppose that  $\mathcal{E}_{ki} \oplus \mathcal{E}_{ik} \neq 0$ .

For 2°, if  $\mathcal{E}_{kj} \oplus \mathcal{E}_{jk} = 0$ , by Proposition 4.3.1, there is a positive integer  $\delta$ , such that

- Either  $\mathcal{E}_{ki} \oplus \mathcal{E}_{ik} = \mathcal{E}_{ki}^\delta$ ,  $\mathcal{E}_{kj} \oplus \mathcal{E}_{jk} = \mathcal{E}_{kj}^{\delta-1}$ , and  $\mathcal{E}_{ki}^\delta = \mathcal{E}_{kj}^{\delta-1} \otimes \mathcal{E}_j i^1$ .
- Or  $\mathcal{E}_{ki} \oplus \mathcal{E}_{ik} = \mathcal{E}_{ik}^\delta$ ,  $\mathcal{E}_{kj} \oplus \mathcal{E}_{jk} = \mathcal{E}_{jk}^{\delta+1}$ , and  $\mathcal{E}_{jk}^{\delta+1} = \mathcal{E}_j i^1 \otimes \mathcal{E}_{ik}^\delta$ .

In the first case,  $(\mathcal{E}^\#)_{jk} = 0$  since  $\mathcal{E}_{ik} = \mathcal{E}_{jk} = 0$ . Applying  $\text{Hom}(S_k, ?)$  to the triangle  $S_i \otimes V_j \rightarrow T_j \rightarrow S_j$ , we get a long exact sequence:

$$\begin{aligned} \rightarrow \text{Hom}^{m-1}(S_k, T_j) \rightarrow \text{Hom}^{m-1}(S_k, S_j) \xrightarrow{f_m} \text{Hom}^m(S_k, S_i \otimes V_j) \rightarrow \\ \rightarrow \text{Hom}^m(S_k, T_j) \rightarrow \dots \end{aligned}$$

Then  $\text{Hom}^{m-1}(S_k, S_j) = \mathcal{E}_{kj}^{m-1} = 0$  and  $\text{Hom}^m(S_k, S_i) = \mathcal{E}_{ki}^m = 0$  except for  $m = \delta$ , in which case  $f_m$  is an isomorphism. Hence  $\text{Hom}^m(S_k, T_j) = (\mathcal{E}^\#)_{kj}^m = 0$  for  $\forall m \in \mathbb{Z}$ . Similarly, we have  $(\mathcal{E}^\#)_{kj} = (\mathcal{E}^\#)_{jk} = 0$  in the second case.

For 3°, we still get the long exact sequence above, but  $\text{Hom}^m(S_k, S_j) = \mathcal{E}_{kj}^m = 0$  for all  $m$ . Hence

$$\text{Hom}^m(S_k, T_j) = \text{Hom}^m(S_k, S_j) \otimes \text{Hom}^1(S_j, S_i \otimes V_j),$$

i.e.  $(\mathcal{E}^\sharp)_{kj}^m = (\mathcal{E}_{ki})^m \otimes (\mathcal{E}_{ji}^1)^*$ . For the same reason  $(\mathcal{E}^\sharp)_{jk}^m = \mathcal{E}_{ji}^1 \otimes \mathcal{E}_{ik}^m$ . □

We also have the corresponding result for backward simple tilting.

**Theorem 4.3.3.** *Let  $\mathcal{H}$  be a heart in  $\text{EG}(Q)$ ,  $S_i \in \text{Sim } \mathcal{H}$ . Denote by  $\mathcal{E}$  and  $\mathcal{E}^\sharp$  the heart algebra of  $\mathcal{H}$  and  $\mathcal{H}_{S_i}^\sharp$  respectively. Define*

$$J_i^\flat = \{j \mid \mathcal{E}_{ij}^1 \neq 0\}, \quad K_i^\sharp = \{j \mid \mathcal{E}_{ij}^1 = 0, j \neq i\}.$$

For  $j \in J_i^\flat, k \in K_i^\sharp, \forall \delta \in \mathbb{Z}$ , We have

$$1^\circ. (\mathcal{E}^\flat)_{ji}^1 = \mathcal{E}_{ij}^1, \quad (\mathcal{E}^\flat)_{ik} = \mathcal{E}_{ik}[1], \quad (\mathcal{E}^\flat)_{ki} = \mathcal{E}_{ki}[-1].$$

$$2^\circ. \text{ If } \mathcal{E}_{ki} \oplus \mathcal{E}_{ik} \neq 0 \text{ and } \mathcal{E}_{kj} \oplus \mathcal{E}_{jk} \neq 0, \text{ then } (\mathcal{E}^\flat)_{kj} = (\mathcal{E}^\flat)_{jk} = 0.$$

$$3^\circ. \text{ If } \mathcal{E}_{ki} \oplus \mathcal{E}_{ik} \neq 0 \text{ and } \mathcal{E}_{kj} = \mathcal{E}_{jk} = 0, \text{ then}$$

$$(\mathcal{E}^\flat)_{kj}^\delta = (\mathcal{E}_{ki})^\delta \otimes (\mathcal{E}_{ji}^1)^*, \quad (\mathcal{E}^\flat)_{jk}^\delta = \mathcal{E}_{ji} \otimes \mathcal{E}_{ik}^\delta.$$

## CHAPTER 5

### MAIN RESULTS: ACYCLIC CASE

Throughout this chapter, let  $Q$  be an acyclic quiver with  $n$  vertices. Recall that we have canonical hearts  $\mathcal{H}_Q$  and  $\mathcal{H}_\Gamma$  in  $\mathcal{D}(Q)$  and  $\mathcal{D}(\Gamma_N Q)$ , respectively, and we will study the exchange graphs  $\text{EG}^\circ(Q)$  and  $\text{EG}^\circ(\Gamma_N Q)$  (which are principal components, c.f. Section 2.4 and Section 2.7).

### 5.1 Inducing hearts

The natural quotient morphism  $\Gamma_N Q \rightarrow Q$  induces a functor

$$\mathcal{I} : \mathcal{D}(Q) \rightarrow \mathcal{D}(\Gamma_N Q). \quad (5.1.1)$$

For more general dg algebras, this functor was considered by Keller, who showed ([34, Lemma 4.4 (b)]) that  $\mathcal{I}$  is a strong L-immersion (see Definition 3.3.1).

Consider the subgraph  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  in  $\text{EG}^\circ(\Gamma_N Q)$  with standard heart  $\mathcal{H}_\Gamma$  as base. Observe that  $\mathcal{I}$  sends the simples in  $\mathcal{H}_Q$  to the corresponding simples in  $\mathcal{H}_\Gamma$  and hence we have  $\mathcal{I}_*(\mathcal{H}_Q) = \mathcal{H}_\Gamma$ .

**Theorem 5.1.1.** *Any heart in  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  induces a heart in  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  via the natural L-immersion  $\mathcal{I}$  in (5.1.1), i.e. we have a well-defined map*

$$\mathcal{I}_* : \text{EG}_N^\circ(Q, \mathcal{H}_Q) \xrightarrow{\sim} \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma). \quad (5.1.2)$$

Moreover, it is an isomorphism between oriented graphs and can be extended to an isomorphism  $\overline{\mathcal{I}}_* : \overline{\text{EG}}_N^\circ(Q, \mathcal{H}_Q) \xrightarrow{\sim} \overline{\text{EG}}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ .

*Proof.* To prove well-definedness of  $\mathcal{I}_*$ , we use induction starting from  $\mathcal{I}_*(\mathcal{H}_Q[1]) = \mathcal{H}_\Gamma[1]$ . Thus, if  $\mathcal{I}_*(\widehat{\mathcal{H}}) = \mathcal{H}$ , for some  $\widehat{\mathcal{H}}, \widehat{\mathcal{H}}_\# \in \text{EG}(Q, \mathcal{H}_Q)$ ,  $\widehat{S} \in \text{Sim } \widehat{\mathcal{H}}$  and  $\mathcal{H} \in \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ , then we need to show that  $\widehat{\mathcal{H}}_\#$  induces a heart in  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ .



For any  $\widehat{X} \in \text{Sim } \widehat{\mathcal{H}}$ , by Proposition 2.2.5 and the first part of Lemma 3.2.2, we know that  $\widehat{X} \in \text{Ind } \mathcal{H}_Q[m]$  for some  $1 \leq m(\widehat{X}) \leq N-1$ . By the second part of Lemma 3.2.2, we have  $\mathbf{H}_{N-1}(\widehat{S}) = 0$  which implies  $m(\widehat{S}) \leq N-2$ , where the homology  $\mathbf{H}_\bullet$  is with respect to  $\mathcal{H}_Q$ . Then  $\text{Hom}^{N-1}(\widehat{S}, \widehat{X}) = 0$ , since  $\mathbf{k}Q$  is hereditary. By Proposition 3.3.3 we have  $F_*(\widehat{\mathcal{H}}_S^\#) = \mathcal{H}_S^\#$ . By the first part of Lemma 3.2.2, we know that  $\mathcal{H}_S^\#$  is in  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ .

The injectivity of  $\mathcal{I}_*$  follows from the facts that a heart is determined by its simples and  $\mathcal{I}$  is injective.

For surjectivity of  $\mathcal{I}_*$ , we consider the intervals. By the first part of Proposition 3.2.2, any interval in  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  has length less or equal than  $N-1$ . Notice that, by Proposition 3.2.4, any maximal interval in  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  has length  $N-1$ , and hence its image under  $\mathcal{I}_*$  is a maximal interval in  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ . This implies that, if a heart  $\mathcal{H}$  in  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  is induced from some heart  $\widehat{\mathcal{H}} \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)$  via  $\mathcal{I}$ , then the maximal interval  $l(\mathcal{H}, S) \cap \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  is induced from the interval  $l(\widehat{\mathcal{H}}, \widehat{S}) \cap \text{EG}_N^\circ(Q, \mathcal{H}_Q)$  via  $\mathcal{I}$ , where  $S \in \text{Sim } \mathcal{H}$ , and  $\widehat{S} \in \text{Sim } \widehat{\mathcal{H}}$  such that  $\mathcal{I}(\widehat{S}) = S$ . Hence any simple tilt of an induced heart via  $\mathcal{I}$  is also induced via  $\mathcal{I}$ , provided this tilt is still in  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ . Thus, inductively, we deduce that  $\mathcal{I}_*$  is surjective.

The last assertion follows from the facts that we can cyclic complete  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  (Proposition 3.2.4) and  $\mathcal{I}_*$  preserves the structure of intervals.  $\square$

**Proposition 5.1.2.**  $\text{Br}(\Gamma_N Q) \cdot \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma) = \text{EG}^\circ(\Gamma_N Q)$ .

*Proof.* We use induction starting from Theorem 5.1.1. Suppose  $\mathcal{H}' \in \text{EG}^\circ(\Gamma_N Q)$  such that  $\mathcal{H}' = \phi(\mathcal{H})$  for  $\phi \in \text{Br}(\Gamma_N Q)$  and  $\mathcal{H} \in \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ . Choose any simple  $S' \in \text{Sim } \mathcal{H}'$  and let  $S = \phi^{-1}(S')$ .

If  $\mathcal{H}_S^\#$  is still in  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ , then by Theorem 5.1.1 we have  $(\mathcal{H}')_{S'}^\# = \phi(\mathcal{H}_S^\#)$ .

Now suppose that  $\mathcal{H}_S^\# \notin \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ . By Theorem 5.1.1, the maximal interval  $l(\mathcal{H}, S) \cap \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  is induced from  $l(\widehat{\mathcal{H}}, \widehat{S}) \cap \text{EG}_N^\circ(Q, \mathcal{H}_Q)$ , where  $\mathcal{H} = \mathcal{I}_*(\widehat{\mathcal{H}})$  and  $\mathcal{I}(\widehat{S}) = S$ . By Proposition 3.2.4, we know these maximal intervals are  $\{\mathcal{H}_S^{mb}\}_{m=0}^{N-2}$  and  $\{\widehat{\mathcal{H}}_{\widehat{S}}^{mb}\}_{m=0}^{N-2}$ . Write  $\mathcal{H}^- = \mathcal{H}_S^{(N-2)b}$  and  $S^- = S[2-N]$ . Since  $\mathcal{H}_S^{mb}$  is induced, then it is good by Proposition 3.3.3, and in particular finite and monochromatic, for  $0 \leq m \leq N-2$ . Applying Corollary 3.1.3 to the simple forward tilt of  $\mathcal{H}_S^{mb}$  with respect to  $S[-m]$ , for  $m = N-2, N-1, \dots, 0$ , we deduce that the changes of simples from  $\mathcal{H}^-$  to  $\mathcal{H}_S^\#$  are as follows:

- for  $S^- \in \text{Sim } \mathcal{H}^-$ , it becomes  $S[1]$  which equals  $\phi_S^{-1}(S^-)$ ;
- for  $X \in \text{Sim } \mathcal{H}^-$  such that  $\text{Hom}^\bullet(X, S) = 0$ , it remains in  $\mathcal{H}_S^\#$ . Observe that  $X = \phi_S^{-1}(X)$ .
- for  $X \in \text{Sim } \mathcal{H}^-$  such that  $\text{Hom}^\bullet(X, S) \neq 0$ , the monochromaticity of  $\mathcal{H}^-$  implies  $\text{Hom}^\bullet(X, S) = \text{Hom}^m(X, S^-)$ , for some integer  $m$ . Notice that  $\mathcal{H}^-$  is induced

from  $\widehat{\mathcal{H}}_{\widehat{S}}^{(N-2)\flat}$ , let  $\widehat{X}$  and  $\widehat{S}^-$  be the corresponding simples in  $\text{Sim } \widehat{\mathcal{H}}_{\widehat{S}}^{(N-2)\flat}$ . By Proposition 2.2.5,  $\widehat{X} \in \mathcal{H}_Q[x]$ ,  $\widehat{S}^- \in \mathcal{H}_Q[s]$  for some integer  $1 \leq x, s \leq N-1$ . Since  $\mathbf{k}Q$  is hereditary, we know that

$$\text{Hom}^{\geq N}(\widehat{X}, \widehat{S}^-) = \text{Hom}^{\geq N}(\widehat{S}^-, \widehat{X}) = 0$$

By (3.3.1), this implies  $1 \leq m \leq N-1$ . Then  $X$  is in  $\text{Sim}(\mathcal{H}^-)_{S^-}^{j\sharp}$  for  $j = 0, \dots, m-1$ , and becomes  $\phi_S^{-1}(X)$  in  $\text{Sim}(\mathcal{H}^-)_{S^-}^{j\sharp}$  for  $j = m, \dots, N-1$ .

Since the simples determine the heart, we have  $\mathcal{H}_S^\sharp = \phi_S^{-1}(\mathcal{H}^-)$  which implies

$$(\mathcal{H}')_{S'}^\sharp = \phi(\mathcal{H}_S^\sharp) = \phi \circ \phi_S^{-1}(\mathcal{H}^-)$$

as required.  $\square$

**Corollary 5.1.3.** *Every heart in  $\text{EG}^\circ(\Gamma_N Q)$  is induced and hence good. Moreover, for any heart  $\mathcal{H}$  in  $\text{EG}^\circ(\Gamma_N Q)$ , the set of twist functors of its simples is a set of generators of  $\text{Br}(\Gamma_N Q)$ . Further, for any  $S \in \text{Sim } \mathcal{H}$ , we have*

$$\mathcal{H}_S^{\pm(N-1)\sharp} = \phi_S^{\mp 1}(\mathcal{H}). \quad (5.1.3)$$

*Proof.* Proposition 5.1.2 shows that every heart is induced via the L-immersion which is the composition of the natural L-immersion  $\mathcal{I}$  with some twist functors. Then every heart is good by Proposition 3.3.3.

Moreover, Corollary 3.1.3 applies to any good heart. Hence the new simples in any simple tilt of such a heart are either the shift or the twist of the old simples. Thus the second assertion follows inductively by (2.8.3).

Further, we know that (5.1.3) is true for any heart  $\mathcal{H}^- \in \mathcal{I}_*(\text{EG}_N^\circ(Q, \mathcal{H}_Q))$  with simple  $S^-$  as in Proposition 5.1.2. Hence it is true for any hearts in  $l(\mathcal{H}^-, S^-)$ , which implies it is also true for any heart induced via  $\mathcal{I}_*$ , by Proposition 3.2.4. Notice that the autoequivalences preserve (5.1.3), thus this equation holds for any heart in  $\text{EG}^\circ(\Gamma_N Q)$  by Proposition 5.1.2.  $\square$

**Corollary 5.1.4.** *Let  $\mathcal{H}$  and  $\mathcal{H}'$  be hearts in  $\text{EG}^\circ(\Gamma_N Q)$  in the same braid group orbit, i.e.  $\phi(\mathcal{H}) = \mathcal{H}'$  for some  $\phi \in \text{Br}(\Gamma_N Q)$ . Then there exists a sequence of spherical objects  $T_0, \dots, T_{m-1}$  in hearts  $\mathcal{H}_0, \dots, \mathcal{H}_{m-1}$  (for some integer  $m \geq 0$ ) together with signs  $\epsilon_i \in \{\pm 1\}$ ,  $i = 0, \dots, m-1$ , such that  $\mathcal{H}_0 = \mathcal{H}$ ,*

$$\mathcal{H}_{i+1} = (\mathcal{H}_i)_{T_i}^{\epsilon_i(N-1)\sharp}, \quad i = 0, 1, \dots, m-1, \quad (5.1.4)$$

and  $\mathcal{H}_m = \mathcal{H}'$ .

*Proof.* Fix  $\mathcal{H}$  and let  $\text{Sim } \mathcal{H} = \{S_1, \dots, S_n\}$ ,  $\phi_k = \phi_{S_k}$  for  $1 \leq k \leq n$ . Since  $\phi_1, \dots, \phi_n$  generate  $\text{Br}(\Gamma_N Q)$  by Corollary 5.1.3, we have

$$\phi = \phi_{t_{m-1}}^{\lambda_{m-1}} \circ \dots \circ \phi_{t_0}^{\lambda_0}$$

for some  $t_j \in \{1, \dots, n\}$  and  $\lambda_j \in \{\pm 1\}$ . Use induction on  $m$ . If  $m = 0$ , i.e.  $\mathcal{H} = \mathcal{H}'$ , there is nothing to prove. Suppose the statement holds for  $m \leq s$  and consider the case when  $m = s + 1$ . Write  $\varphi = \phi_{t_s}^{\lambda_s}$ . For hearts  $\mathcal{H}$  and

$$\varphi^{-1}(\mathcal{H}') = \left( \phi_{t_{s-1}}^{\lambda_{s-1}} \circ \dots \circ \phi_{t_0}^{\lambda_0} \right) (\mathcal{H}),$$

by inductive hypothesis, there are spherical objects  $R_0, R_2, \dots, R_{s-1}$  together with  $\varepsilon_i \in \{\pm 1\}$ , such that  $\mathcal{H}'_0 = \mathcal{H}$ ,

$$\mathcal{H}'_{i+1} = (\mathcal{H}'_i)_{R_i}^{\varepsilon_i(N-1)\sharp}, \quad i = 0, 1, \dots, s-1$$

and  $\mathcal{H}'_s = \varphi^{-1}(\mathcal{H}')$ . Let  $T_0 = S_{t_m}$ ,  $\epsilon_0 = \lambda_m$  and  $T_i = \varphi(R_{i-1})$ ,  $\epsilon_i = \varepsilon_{i-1}$  for  $i = 1, \dots, s$ . Then we have  $\mathcal{H}_0 = \mathcal{H}$ ,  $\mathcal{H}_1 = \varphi(\mathcal{H}_0)$  and (inductively)

$$\mathcal{H}_{i+1} = (\mathcal{H}_i)_{T_i}^{\epsilon_i(N-1)\sharp} = \left( \varphi(\mathcal{H}'_{i-1}) \right)_{\varphi(R_{i-1})}^{\varepsilon_{i-1}(N-1)\sharp} = \varphi(\mathcal{H}'_i)$$

for  $i = 1, \dots, s$ . In particular, we have  $\mathcal{H}_{s+1} = \varphi(\mathcal{H}'_s) = \mathcal{H}'$  as required.  $\square$

## 5.2 Cyclically completing

By Proposition 5.1.2, there is a surjection on vertex sets

$$p_0 : \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma) \twoheadrightarrow \text{EG}^\circ(\Gamma_N Q) / \text{Br}.$$

Moreover, by the proof of Proposition 5.1.2, we can extend  $p_0$  to a surjection (between oriented graphs)

$$\overline{p_0} : \overline{\text{EG}_N^\circ}(\Gamma_N Q, \mathcal{H}_\Gamma) \twoheadrightarrow \text{EG}^\circ(\Gamma_N Q) / \text{Br}.$$

sending the new edge  $e_l$  in each basic cycle  $c_l$  to the edge in  $\text{EG}^\circ(\Gamma_N Q) / \text{Br}$  induced by  $(\mathcal{H} \xrightarrow{S} \mathcal{H}_S^\sharp)$ , where  $c_l$  is induced by the line  $l = l(\mathcal{H}, S)$  such that

$$l(\mathcal{H}, S) \cap \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma) = \{\mathcal{H}_S^{\sharp b}\}_{i=0}^{N-2}. \quad (5.2.1)$$

**Theorem 5.2.1.** *Let  $Q$  be an acyclic quiver. As oriented exchange graphs, we have a canonical isomorphism*

$$\overline{p_0} : \overline{\text{EG}_N^\circ}(\Gamma_N Q, \mathcal{H}_\Gamma) \cong \text{EG}^\circ(\Gamma_N Q) / \text{Br}(\Gamma_N Q), \quad (5.2.2)$$

and hence

$$\mathrm{EG}^\circ(\Gamma_N Q)/\mathrm{Br}(\Gamma_N Q) \cong \mathrm{CEG}_{N-1}(Q). \quad (5.2.3)$$

*Proof.* There is an exact sequence of triangulated categories (c.f. [1], also [29])

$$0 \rightarrow \mathcal{D}(\Gamma_N Q) \rightarrow \mathrm{per}(\Gamma_N Q) \rightarrow \mathcal{C}_{N-1}(Q) \rightarrow 0,$$

where  $\mathrm{per}(\Gamma_N Q)$  is the perfect derived category of  $\Gamma_N Q$ . By [1, Section 2], every heart  $\mathcal{H}$  in  $\mathrm{EG}^\circ(\Gamma_N Q)$  induces a t-structure on  $\mathrm{per}(\Gamma_N Q)$  and determines a silting object in  $\mathrm{per}(\Gamma_N Q)$ , which induces a tilting object in  $\mathcal{C}_{N-1}(Q)$ . Thus we have a map  $v : \mathrm{EG}^\circ(\Gamma_N Q) \rightarrow \mathrm{CEG}_{N-1}(Q)$ . Moreover, via  $v$ ,  $\mathcal{H}_\Gamma$  corresponds to the initial cluster tilting set and the simple tilting of a heart corresponds to the mutation of a tilting/silting object.

By Corollary 5.1.4, if two hearts  $\mathcal{H}, \mathcal{H}' \in \mathrm{EG}^\circ(\Gamma_N Q)$  are in the same braid group orbit, then  $\mathcal{H}'$  can be obtained from  $\mathcal{H}$  by a sequence of simple tiltings as in (5.1.4). Then  $v(\mathcal{H}) = v(\mathcal{H}')$  because repeating the same mutation  $N - 1$  times returns every cluster tilting object back to itself. Hence we have a map  $\bar{v} : \mathrm{EG}^\circ(\Gamma_N Q)/\mathrm{Br} \rightarrow \mathrm{CEG}_{N-1}(Q)$ .

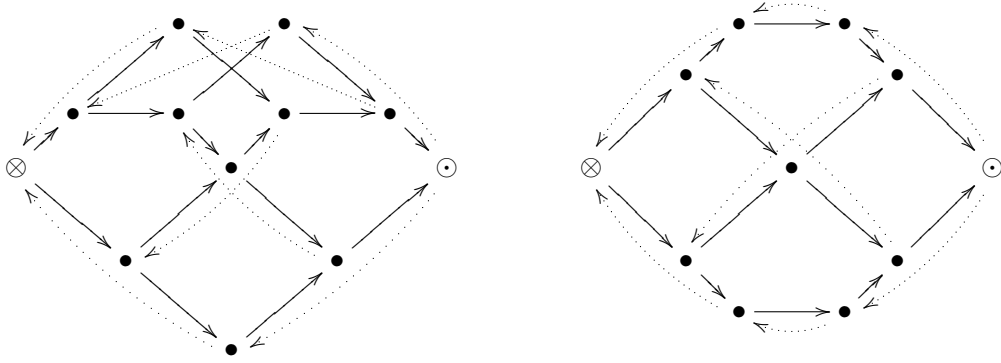
Inductively, we know that the simple tilting in  $\mathrm{EG}_N^\circ(Q, \mathcal{H}_Q)$  or  $\mathrm{EG}^\circ(\Gamma_N Q)$  corresponds to the mutation of a tilting/silting object. Thus we obtain the following commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{EG}}_N^\circ(Q, \mathcal{H}_Q) & \xrightarrow[\cong]{\bar{\mathcal{T}}} & \mathrm{CEG}_{N-1}(Q) \\ \cong \downarrow \bar{\mathcal{T}}_* & & \uparrow \bar{v} \\ \overline{\mathrm{EG}}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma) & \xrightarrow{\bar{p}_0} & \mathrm{EG}^\circ(\Gamma_N Q)/\mathrm{Br} \end{array} \quad (5.2.4)$$

which implies the theorem.  $\square$

**Remark 5.2.2.** We need the standard heart as base on the left-hand-side to ensure the isomorphism (5.2.2) holds. Example 5.2.3 illustrates this phenomenon. However, if  $N = 3$ , isomorphism (5.2.2) holds for any heart (see Section 5.4). Further, for  $N = 3$ , Keller-Nicolás (c.f. [32, Theorem 5.6]) proves (5.2.3) in full generality, that is, when  $Q$  is a loop-free and 2-cycle-free quiver with a polynomial potential  $W$ .

**Example 5.2.3.** Let  $Q$  be a quiver of type  $A_2$  with corresponding  $\mathrm{Sim} \mathcal{H}_\Gamma = \{S, X\}$  such that  $\mathrm{Hom}^1(S, X) = \mathbf{k}$ . Figure 5-1 shows the cyclic completions of two exchange graphs:  $\overline{\mathrm{EG}}_4(\Gamma_N Q, \mathcal{H}_\Gamma)$  on the left and  $\overline{\mathrm{EG}}_4(\Gamma_N Q, (\mathcal{H}_\Gamma)_S^\sharp)$  on the right. The solid arrows are the edges in  $\mathrm{EG}^\circ(\Gamma_N Q)$  and the dotted arrows are the extra edges in the cyclic completions. The vertices  $\otimes$  and  $\odot$  represent the source and sink (i.e.  $\mathcal{H}[1]$  and  $\mathcal{H}[3]$  in fact) in the exchange graph  $\mathrm{EG}_4(\Gamma_4 Q, \mathcal{H})$  with base  $\mathcal{H}$ . Notice that



**Figure 5-1:** Two cyclic completions of CY-4 exchange graphs of  $A_2$ -type

$$\overline{\text{EG}}_4(\Gamma_N Q, (\mathcal{H}_\Gamma)^\sharp_S) \neq \text{EG}^\circ(\Gamma_N Q)/\text{Br}.$$

**Remark 5.2.4.** By Corollary 3.2.5, each almost complete cluster tilting set in  $\mathcal{C}_{N-1}(Q)$  can be identified with a basic cycle in  $\text{CEG}_{N-1}(Q) \cong \overline{\text{EG}}_N^\circ(Q, \mathcal{H}_Q)$ , which can be identified with a basic cycle in  $\overline{\text{EG}}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  by Theorem 5.1.1. By Theorem 5.2.1, these basic cycles also can be interpreted as braid group orbits of lines of  $\text{EG}^\circ(\Gamma_N Q)$  in  $\text{EG}^\circ(\Gamma_N Q)/\text{Br}$ .

### 5.3 A limit formula for exchange graphs

**Proposition 5.3.1.** *We have.*

$$\text{EG}^\circ(Q) = \lim_{N \rightarrow \infty} \text{EG}_{2N}(Q, \mathcal{H}_Q[-N]). \quad (5.3.1)$$

Besides,  $\text{EG}_N(Q, \mathcal{H}_Q)$  is finite if  $Q$  is of Dynkin type.

*Proof.* Let  $\mathcal{H} \in \text{EG}(Q)$ . Consider the homology  $\mathbf{H}_\bullet$ , with respect to  $\mathcal{H}_Q$ , of any simple  $S$  of  $\mathcal{H}$ . Then we know that if  $N \gg 1$ , then  $S \in \bigcup_{j=1-N}^{N-1} \mathcal{H}_Q[j]$  which implies  $\mathcal{H}_Q[-N+1] \leq \mathcal{H} \leq \mathcal{H}_Q[N-1]$ . Then  $\mathcal{H} \in \text{EG}_{2N}(Q, \mathcal{H}_Q[-N])$  which implies (5.3.1).

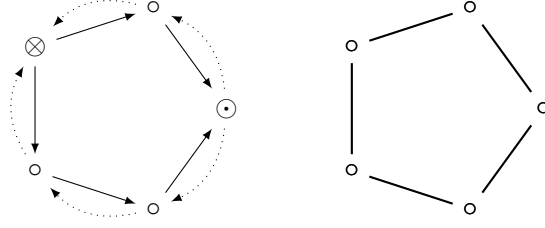
Notice that there are only finitely many indecomposables in  $\bigcup_{j=1}^{N-1} \mathcal{H}_Q[j]$  for  $Q$  of Dynkin type and hence only finitely many hearts in  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$ .  $\square$

Therefore, we have the following limit formula.

**Corollary 5.3.2.** *We have*

$$\text{EG}^\circ(Q) \cong \lim_{N \rightarrow \infty} \text{EG}^\circ(\Gamma_N Q)/\text{Br}(\Gamma_N Q)$$

in the following sense:



**Figure 5-2:**  $\text{EG}(\Gamma_3 Q)/\text{Br}_3$  and  $\text{CEG}(Q)^*$  for an  $A_2$ -type quiver  $Q$

1°. There exists a family of subgraphs  $\{E_N(Q)\}_{N \geq 2}$  in  $\text{EG}(Q)$  satisfying  $E_N(Q) \subset E_{N+1}(Q)$  and  $\text{EG}^\circ(Q) \cong \lim_{N \rightarrow \infty} E_N(Q)$ .

2°.  $E_N(Q)$  is isomorphic to a fundamental domain for  $\text{EG}^\circ(\Gamma_N Q)/\text{Br}$ .

*Proof.* Take  $E_N(Q) = \text{EG}_N^\circ(Q, \mathcal{H}_Q[m])$ , where  $m = \lfloor -\frac{N}{2} \rfloor$ . The first condition follows from Proposition 5.3.1. Notice that  $E_N(Q) \cong \text{EG}_N^\circ(Q, \mathcal{H}_Q)$ , the second condition follows from Theorem 5.1.1.  $\square$

## 5.4 Orientations of cluster exchange graphs

In this subsection, we fix  $N = 3$ . In  $\text{EG}^\circ(\Gamma_3 Q)/\text{Br}_3$ , each cycle is a 2-cycle, which induces nice properties in the following sense.

- $\text{EG}^\circ(\Gamma_3 Q)/\text{Br}_3 \cong \overline{\text{EG}}_3^\circ(\Gamma_3 Q, \mathcal{H})$  for any heart  $\mathcal{H} \in \text{EG}^\circ(\Gamma_3 Q)$  (which is not true for  $N > 3$ , as in Example 5.2.3).
- The underlying graph of  $\text{EG}(\Gamma_3 Q, \mathcal{H})$ , for any  $\mathcal{H} \in \text{EG}^\circ(\Gamma_3 Q)$ , is the usual (unoriented) cluster exchange graph  $\text{CEG}(Q)^*$ , that is. the graph obtained from  $\text{CEG}_2(Q)$  by replacing each basic (2-)cycle with an unoriented edge.

For example, for  $Q$  of type  $A_2$ , we have the oriented exchange graph  $\text{EG}(\Gamma_3 Q)/\text{Br}_3$  on the left of Figure 5-2, which is constructed as the cyclic completion  $\overline{\text{EG}}_3^\circ(\Gamma_3 Q, \mathcal{H})$  for any  $\mathcal{H} \in \text{EG}(\Gamma_3 Q)$ . On the right of Figure 5-2, We have the corresponding unoriented cluster exchange graph  $\text{CEG}(Q)^*$ . The rest of this subsection is devoted to explain these properties of the CY-3 case in detail.

**Lemma 5.4.1.** *Let  $\mathcal{H} \in \text{EG}(\mathcal{D})$  for some triangulated category  $\mathcal{D}$  and  $\mathcal{H}[1] \leq \mathcal{H}_0 \leq \mathcal{H}[2]$ . Choose  $S \in \text{Sim } \mathcal{H}$  such that the simple tilts of  $\mathcal{H}$  with respect to  $S$  exist.*

1°. *If  $S[1] \notin \mathcal{H}_0$ , then  $\mathcal{H}_S^\sharp[1] \leq \mathcal{H}_0$ .*

2°. *If  $S[2] \notin \mathcal{H}_0$ , then  $\mathcal{H}_0 \leq \mathcal{H}_S^\flat[2]$ .*

*Proof.* For 1°, note first that  $S[1] \in \mathcal{P}^\perp[2] \subset \mathcal{P}_0^\perp[1]$ , so that  $S[1] \notin \mathcal{H}_0$  implies  $S[1] \notin \mathcal{P}_0$ . Now, suppose that  $M \notin \mathcal{P}_S^\sharp[1]$  but  $M \in \mathcal{P}_0$ . Consider the filtrations (2.3.1) and

(2.3.2) of  $M$ , with respect to  $\mathcal{H}$ , and the torsion pair corresponding to  $\mathcal{H}_S^\sharp$ . Since  $M \in \mathcal{P}_0 \subset \mathcal{P}[1]$ , we have  $k_m \geq 1$ . But  $M \notin \mathcal{P}_S^\sharp[1]$  forces  $k = 1$  and  $H_m^{\mathcal{F}} = S^t \neq 0$ . In this case, there is a triangle  $M' \rightarrow M \rightarrow S[1] \rightarrow M'[1]$  with  $M' \in \mathcal{P}[1]$ . Then we have  $M'[1] \in \mathcal{P}[2] \subset \mathcal{P}_0$  and  $M \in \mathcal{P}_0$ , which implies  $S[1] \in \mathcal{P}_0$ , contradicting to  $S[1] \notin \mathcal{P}_0$ . Thus  $\mathcal{P}_S^\sharp[1] \supset \mathcal{P}_0$ , i.e.  $\mathcal{H}_S^\sharp[1] \leq \mathcal{H}_0$ .

Similarly for  $2^\circ$ , we have  $S[2] \notin \mathcal{P}_0^\perp[1]$ . If there is an object  $M \notin \mathcal{P}_S^\flat[2]^\perp$  but  $M \in \mathcal{P}_0^\perp$ , we deduce as before that  $k_1 = 1$  with  $H_1^{\mathcal{T}} = S^t \neq 0$  in (2.3.2). Then there is a triangle  $M'[-1] \rightarrow S^t[1] \rightarrow M \rightarrow M'$  with  $M' \in \mathcal{P}^\perp[1] \subset \mathcal{P}_0^\perp$ . Then we have  $S[1] \in \mathcal{P}_0^\perp$  contradicting to  $S[2] \notin \mathcal{P}_0^\perp[1]$ . Thus  $\mathcal{P}_S^\flat[2]^\perp \supset \mathcal{P}_0^\perp$ , i.e.  $(\mathcal{H}_S^\flat)^\perp[2] \leq \mathcal{H}_0^\perp$ .  $\square$

For  $S \in \text{Sim } \mathcal{H}$ , we define

$$\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^- = \{\mathcal{H}' \in \text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H}) \mid S[1] \in \mathcal{H}'\} \quad (5.4.1)$$

$$\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^+ = \{\mathcal{H}' \in \text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H}) \mid S[2] \in \mathcal{H}'\}. \quad (5.4.2)$$

**Corollary 5.4.2.** *Let  $\mathcal{H} \in \text{EG}(\mathcal{D})$  for some triangulated category  $\mathcal{D}$  and  $\mathcal{H}[1] \leq \mathcal{H}_0 \leq \mathcal{H}[2]$ . Choose  $S \in \text{Sim } \mathcal{H}$  such that the simple tilts of  $\mathcal{H}$  with respect to  $S$  exist. and we have*

$$S \in \mathcal{H}_0[-1] \sqcup \mathcal{H}_0[-2]. \quad (5.4.3)$$

Hence we have

- $S[2] \in \mathcal{H}_0$  if and only if  $\mathcal{H}_S^\sharp[1] \leq \mathcal{H}_0$ ,
- $S[1] \in \mathcal{H}_0$  if and only if  $\mathcal{H}_0 \leq \mathcal{H}_S^\flat[2]$ ,
- $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^- \cup \text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^+ = \text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$ .

*Proof.* By Lemma 5.4.1, it is enough to show that (5.4.3). If  $S[1] \notin \mathcal{H}_0$ , then we have  $\mathcal{H}_S^\sharp[1] \leq \mathcal{H}_0 \leq \mathcal{H}[2]$  by Lemma 5.4.1. Then we have

$$S[2] \in \mathcal{P}[2] \subset \mathcal{P}_0 \quad (5.4.4)$$

$$S[2] \in \mathcal{P}_S^\sharp[1]^\perp \subset \mathcal{P}_0^\perp[1] \quad (5.4.5)$$

which implies  $S[2] \in \mathcal{H}_0$ . Similarly,  $S[2] \notin \mathcal{H}_0$  implies  $S[1] \in \mathcal{H}_0$ .  $\square$

**Corollary 5.4.3.** *For any  $\mathcal{H} \in \text{EG}^\circ(\Gamma_3 Q)$ ,  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$  has a unique source  $\mathcal{H}[1]$  and a unique sink  $\mathcal{H}[2]$ .*

*Proof.* Let  $\mathcal{H}_0 \in \text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$  with any simple  $S_0 \in \text{Sim } \mathcal{H}_0$ , we have  $\mathcal{H}_0 \leq \mathcal{H}[2] \leq \mathcal{H}_0[1]$ . By (5.4.3) in Corollary 5.4.2, we have  $S_0 \in \mathcal{H}[1] \sqcup \mathcal{H}[2]$ .

Now if  $\mathcal{H}_0$  is a source, we have  $(\mathcal{H}_0)_{S_0}^\sharp \in \text{EG}_3(\Gamma_3 Q, \mathcal{H})$  for any  $S_0 \in \text{Sim } \mathcal{H}_0$ . By the second part of Lemma 3.2.2, we must have  $S_0 \in \mathcal{H}[1]$  instead of  $S_0 \in \mathcal{H}[2]$ . Thus

$\mathcal{H}_0 \subset \mathcal{H}[1]$  which implies  $\mathcal{H}_0 = \mathcal{H}[1]$ , or equivalently,  $\mathcal{H}[1]$  is the unique source. Similar for the uniqueness of the sink.  $\square$

**Lemma 5.4.4.** *Let  $\mathcal{H} \in \text{EG}^\circ(\Gamma_3 Q)$ ,  $S \in \text{Sim } \mathcal{H}$  and  $e$  be an edge connecting  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^\pm$ . Then the tail of  $e$  is in  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^-$  and the label of  $e$  is  $S[1]$ .*

*Proof.* Let  $\mathcal{H}_1 \in \text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^-$  and  $\mathcal{H}_2 \in \text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^+$  be the vertices of  $e$ . Since  $S[1] \in \mathcal{P}_1$  and  $S[1] \notin \mathcal{P}_2$ , we must have  $\mathcal{P}_1 \supset \mathcal{P}_2$ , i.e.  $\mathcal{H}_1$  is the tail of  $e$ . Let  $T$  be the label of  $e$  and  $(\mathcal{F}, \mathcal{T})$  is the torsion pair in  $\mathcal{H}_1$  corresponding to  $e$ . By (5.4.3) we have  $T \in \mathcal{H}[1]$ . Suppose  $T \neq S[1]$ . Noticing  $\mathcal{T} \in \mathcal{H}_2$  but  $S[1] \notin \mathcal{H}_2$ , we have  $S[1] \notin \mathcal{T}$  which implies there is a nonzero map  $f: S[1] \rightarrow T$ . Let  $M = \text{Cone}(f)[-1]$ . Since  $T$  is a simple in  $\mathcal{H}_1$ , we have  $M \in \mathcal{H}_1$ . On the other hand,  $S[1]$  is a simple in  $\mathcal{H}[1]$ , we have  $M \in \mathcal{H}[1]$  contradicting to  $M \in \mathcal{H}_1$ . Hence  $T = S[1]$ .  $\square$

**Proposition 5.4.5.** *For any heart  $\mathcal{H} \in \text{EG}^\circ(\Gamma_3 Q)$  and  $S \in \text{Sim } \mathcal{H}$ , the exchange graph  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H}_S^\#)$  can be obtained from  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$  by applying a ‘half-twist’, that is, applying  $\phi_S^{-1}$  to  $\text{EG}(\Gamma_3 Q, \mathcal{H})_S^-$ , reversing all edges with label  $S[1]$  in  $\text{EG}(\Gamma_3 Q, \mathcal{H})$  and relabeling them with  $S[2]$ .*

*Proof.* First, by Lemma 5.4.1, we have  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^+ = \text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H}_S^\#)_{S[1]}^-$  and

$$\begin{aligned} \phi^{-1}(\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^-) &= \phi^{-1}(\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H}_S^\#)_{S[-1]}^+) \\ &= \text{EG}_3^\circ(\Gamma_3 Q, \phi^{-1}(\mathcal{H}_S^\#))_{\phi^{-1}(S[-1])}^+ = \text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H}_S^\#)_{S[1]}^+. \end{aligned}$$

Second, by Lemma 5.4.4, we know that any edge connecting  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^\pm$  is labeled by  $S[1]$  and any edge connecting  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H}_S^\#)_{S[1]}^\pm$  is labeled by  $S[2]$ . Thus the theorem follows by the fact that the edge  $(\mathcal{H}_1 \xrightarrow{S[1]} \mathcal{H}_2)$  in  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$  becomes the edge  $(\phi_S^{-1}(\mathcal{H}_1) \xleftarrow{S[2]} \mathcal{H}_2)$  in  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H}_S^\#)$  under the half twist.  $\square$

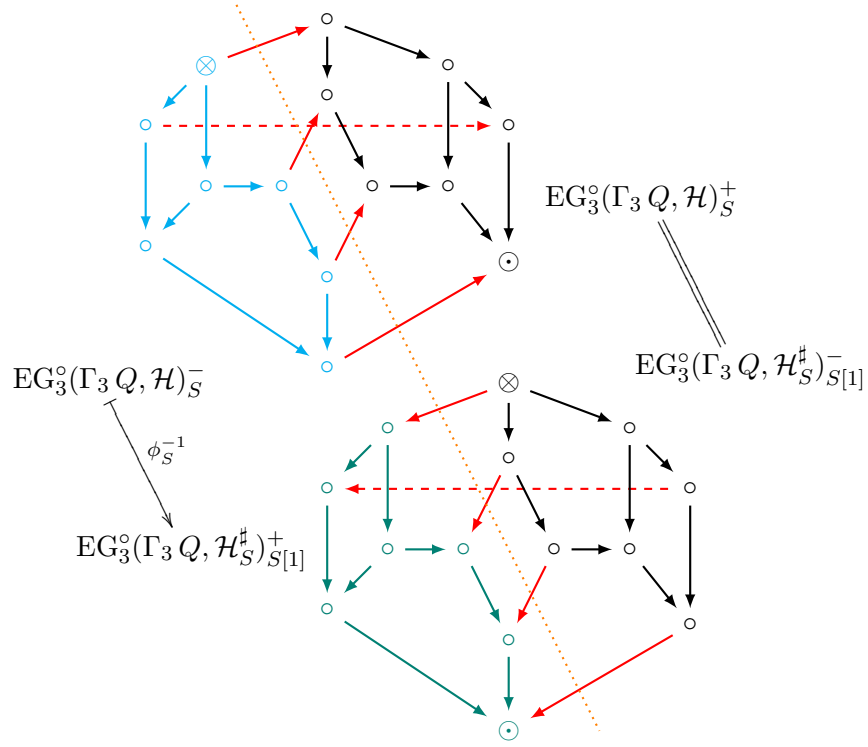
Then, inductively, we have the following result.

**Theorem 5.4.6.** *For any heart  $\mathcal{H} \in \text{EG}^\circ(\Gamma_3 Q)$ ,  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$  is an orientation of the (unoriented) cluster exchange graph  $\text{CEG}(Q)^*$ . Equivalently, we have*

$$\overline{\text{EG}_3^\circ}(\Gamma_3 Q, \mathcal{H}) \cong \text{CEG}_2(Q) \cong \text{EG}^\circ(\Gamma_3 Q) / \text{Br}_3.$$

**Example 5.4.7.** Let  $Q$  be the  $A_3$ -type quiver in Example 2.4.2,  $\mathcal{H} = \mathcal{H}_\Gamma$  and  $\mathcal{I}(Y) = S$ . In Figure 5-3, the black and blue parts of the top graph are  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})_S^\pm$ ; whereas the green and black parts of the bottom graph are  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H}_S^\#)_{S[1]}^\pm$ . Moreover, the red arrows which connect  $\text{EG}_3^\circ(\Gamma_3 Q, ?)_?^\pm$  are  $S[1]$ -parallel edges in the top graph and  $S[2]$ -parallel edges in the bottom graph. The vertices  $\otimes$  and  $\odot$  are the unique source and sink in the graphs, respectively.





**Figure 5-3:** *Half twist of  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$  for an  $A_3$ -type quiver  $Q$*

**Remark 5.4.8.** Proposition 5.4.5 encodes how the subgraphs  $\text{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$  glue together as in Example 5.4.7. This result was essentially due to the work of Plamondon, Nagao and Keller (c.f. [46], [44], [32]).

## CHAPTER 6

### MAIN RESULTS: DYNKIN CASE

In this chapter, suppose  $Q$  is of Dynkin type and we can identify  $\text{Br}(\Gamma_N Q)$  with the braid group  $\text{Br}_Q$ . We will study the exchange graphs and stability spaces for  $\mathcal{D}(Q)$  and  $\mathcal{D}(\Gamma_N Q)$ .

### 6.1 Simply connectedness of $\text{Stab}(Q)$

Let  $\text{Stab}(Q) = \text{Stab}(\mathcal{D}(Q))$ . The connectedness of  $\text{Stab}(Q)$  follows from the connectedness of  $\text{EG}(Q)$ .

#### 6.1.1 A canonical embedding

By Theorem 4.2.5, we have a disjoint union  $\text{Stab}(Q) = \bigcup_{\mathcal{H} \in \text{EG}(Q)} \text{U}(\mathcal{H})$ . Moreover, by the results in [51, Section 2], we have

$$\overline{\text{U}(\mathcal{H})} - \text{U}(\mathcal{H}) = \bigcup_{\mathcal{H}[-1] \leq \mathcal{H}' < \mathcal{H}} \left( \overline{\text{U}(\mathcal{H})} \cap \text{U}(\mathcal{H}') \right), \quad (6.1.1)$$

and hence the gluing structure of  $\text{Stab}(Q) = \bigcup_{\mathcal{H} \in \text{EG}(Q)} \overline{\text{U}(\mathcal{H})}$  is encoded by the following formula

$$\partial \text{U}(\mathcal{H}) = \bigcup_{\mathcal{H}[-1] \leq \mathcal{H}' < \mathcal{H}} \left( \overline{\text{U}(\mathcal{H})} \cap \text{U}(\mathcal{H}') \right) \cup \bigcup_{\mathcal{H} < \mathcal{H}' \leq \mathcal{H}[1]} \left( \overline{\text{U}(\mathcal{H}')} \cap \text{U}(\mathcal{H}) \right), \quad (6.1.2)$$

Call a term in the RHS in (6.1.2) a face of the  $n$ -cell  $\text{U}(\mathcal{H})$ . Further, by [6, Lemma 5.5], codimension one faces of  $\mathcal{H}$  corresponds to its simple tilts. More precisely,  $\dim \overline{\text{U}(\mathcal{H})} \cap \text{U}(\mathcal{H}) = n - 1$  if and only if  $\mathcal{H}' = \mathcal{H}_S^\sharp$  or  $\mathcal{H}' = \mathcal{H}_S^\flat$  for some  $S \in \text{Sim } \mathcal{H}$ . Therefore, we have the following lemma.

**Lemma 6.1.1.** *There is a canonical embedding (unique up to homotopy)*

$$\iota : \text{EG}(Q) \hookrightarrow \text{Stab}(Q) \quad (6.1.3)$$

such that

1°. for each vertex (heart)  $\mathcal{H}$ , its image is the center of the  $n$ -cell  $U(\mathcal{H})$ , i.e.  $\iota(\mathcal{H}) = (Z_{\mathcal{H}}, \mathcal{P}_{\mathcal{H}})$  with heart  $\mathcal{H}$  satisfying  $Z_{\mathcal{H}}(S_j) = \exp(\frac{1}{2}\pi \mathbf{i})$ .

2°. for each edge  $S_i : \mathcal{H} \rightarrow \mathcal{H}_{S_i}^{\sharp}$ , its image  $\sigma_{(0,1)} = \{\sigma_t = (Z_t, \mathcal{P}_t) \mid t \in (0,1)\}$  is contained in  $(U(\mathcal{H}) \cup U(\mathcal{H}_{S_i}^{\sharp}))^{\circ}$  and intersects  $U(\mathcal{H}) \cap \overline{U(\mathcal{H}_{S_i}^{\sharp})}$  exactly once.

Now we fix a canonical embedding  $\iota$  and will identify the exchange graph with the image of this embedding.

**Lemma 6.1.2.** *We have a surjection  $\pi_1(\text{EG}(Q)) \rightarrow \pi_1(\text{Stab}(Q))$ .*

*Proof.* Let  $Y$  be the union of all faces, with codimension bigger than one, of some heart in  $\text{EG}(Q)$ . We can slightly perturb any path in  $\text{Stab}(Q)$ , without changing its class in  $\pi_1(\text{Stab}(Q))$ , such that it misses  $Y$ . Since  $\text{Stab}^{\circ}(Q) - Y$  contracts onto  $\text{EG}(Q)$ , the lemma follows.  $\square$

### 6.1.2 Simply connectedness

**Lemma 6.1.3.** *Let  $\mathcal{H}$  be a heart in  $\mathcal{D}(Q)$  and  $S_i, S_j$  be two simples in  $\text{Sim } \mathcal{H}$ . Suppose that  $\text{Hom}^1(S_i, S_j) = 0$ . Let  $\mathcal{H}_i = \mathcal{H}_{S_i}^{\sharp}$ ,  $\mathcal{H}_j = \mathcal{H}_{S_j}^{\sharp}$  and  $\mathcal{H}_{ij} = (\mathcal{H}_j)_{S_i}^{\sharp}$ .*

1°. *If  $\text{Hom}^1(S_j, S_i) = 0$ , then  $(\mathcal{H}_i)_{S_j}^{\sharp} = \mathcal{H}_{ij}$ .*

2°. *If  $\text{Hom}^1(S_j, S_i) \neq 0$ , let  $T_j = \phi_{S_i}^{-1}(S_j)$ . Then we have  $\mathcal{H}_{ij} = (\mathcal{H}_*)_{S_j}^{\sharp}$ , where  $\mathcal{H}_* = (\mathcal{H}_i)_{T_j}^{\sharp}$ .*

$$\begin{array}{ccc}
 & \mathcal{H}_i & \\
 S_i \nearrow & & \searrow S_j \\
 \mathcal{H} & & \mathcal{H}_{ij} \\
 S_j \searrow & & \nearrow S_i \\
 & \mathcal{H}_j &
 \end{array}
 \quad
 \begin{array}{ccccc}
 & \mathcal{H}_i & \xrightarrow{T_j} & \mathcal{H}_* & \\
 S_i \nearrow & & & & \searrow S_j \\
 \mathcal{H} & & & & \mathcal{H}_{ij} \\
 S_j \searrow & & & & \nearrow S_i \\
 & \mathcal{H}_j & & &
 \end{array}
 \quad (6.1.4)$$

*Proof.* We have  $\dim \text{Hom}^{\bullet}(S_j, S_i) \leq 1$  by Proposition 4.3.1. Applying Theorem 3.1.4, the lemma follows by a direct calculation.  $\square$

**Proposition 6.1.4.** *If  $Q$  is of Dynkin type, then  $\pi_1(\text{EG}_N(Q, \mathcal{H}_Q))$  is generated by squares and pentagons as in (6.1.4) for any  $N \geq 2$ . Further,  $\pi_1(\text{EG}(Q))$  is generated by such squares and pentagons.*

*Proof.* Choose any cycle  $c$  in  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$ . By Proposition 5.3.1,

$$B(c) = \{\mathcal{H} \mid \exists \mathcal{H}' \in c, \mathcal{H}' \leq \mathcal{H} \leq \mathcal{H}_Q[N-1]\}$$

is finite. We use induction on  $\#B(c)$  to prove the first statement. If  $\#B(c) = 1$ , then  $c$  is trivial. Suppose that  $\#B(c) > 1$  and any cycle  $c' \subset \text{EG}_N^\circ(Q, \mathcal{H}_Q)$  with  $\#B(c') < \#B(c)$  is generated by the squares and pentagons. Choose a source  $\mathcal{H}$  in  $c$  such that  $\mathcal{H}' \not\prec \mathcal{H}$  for any other source  $\mathcal{H}'$  in  $c$ . Let  $S_i$  and  $S_j$  be the arrows coming out at  $\mathcal{H}$ . If  $i = j$  we can delete them in  $c$  to get a new cycle  $c'$ . If  $i \neq j$ , we know that  $S_i : \mathcal{H} \rightarrow \mathcal{H}_i$  and  $S_j : \mathcal{H} \rightarrow \mathcal{H}_j$  are either in a square or a pentagon as in (6.1.4). By the second part of Lemma 3.2.2, we know that  $\mathbf{H}_{N-1}(S_i) = 0$  and hence  $\mathcal{H}_{ij} = (\mathcal{H}_j)_{S_i}^\# \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)$ . Thus this square/pentagon are in  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  and we can replace  $S_i$  and  $S_j$  in  $c$  by other edges in this square/pentagon to get a new cycle  $c' \subset \text{EG}_N^\circ(Q, \mathcal{H}_Q)$ . Either way, we have  $B(c') \subset (B(c) - \{\mathcal{H}\})$  for the new cycle  $c'$  and we are done.

Now choose any cycle  $c$  in  $\text{EG}(Q)$ . By (5.3.1), we can choose  $N \gg 1$  such that all hearts in  $c[k]$  are in  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  for some integer  $k$ . Then the second statement follows from the first one.  $\square$

**Lemma 6.1.5.** *Any square or pentagon as in (6.1.4) is trivial in  $\pi_1(\text{Stab}(Q))$ .*

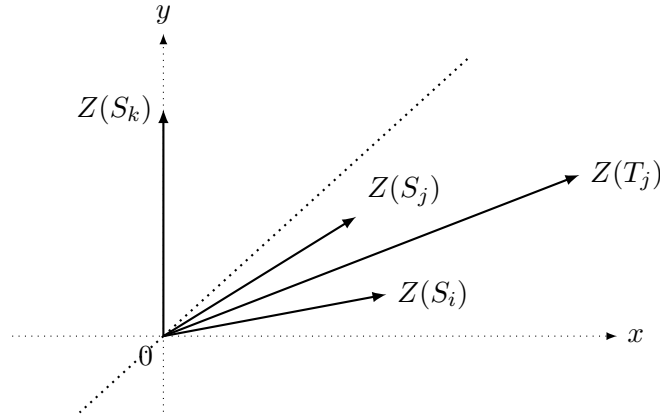


Figure 6-1

*Proof.* Recall that we embed  $\text{EG}(Q)$  into  $\text{Stab}(Q)$ . Suppose in case 2° of Lemma 6.1.3 and consider the path  $L_p : \mathcal{H} \rightarrow \mathcal{H}_i \rightarrow \mathcal{H}_* \rightarrow \mathcal{H}_{ij}$  in  $\text{EG}(Q)$ . Let  $\text{Sim } \mathcal{H} = \{S_1, \dots, S_n\}$ .

Consider the stability condition  $\sigma$  whose heart is  $\mathcal{H}$  satisfying

$$\begin{cases} Z(S_k) = \exp(\frac{1}{2}\pi\mathbf{i}) & k \neq i, j, \\ Z(S_i) = \exp(\delta\pi\mathbf{i}), \\ Z(S_j) = \exp(3\delta\pi\mathbf{i}), \end{cases}$$

for some small  $\delta > 0$ . Notice that,  $\dim \operatorname{Hom}^\bullet(S_j, S_i) = 1$ , hence there are only three indecomposables in  $\mathcal{H}$  generated by  $S_i$  and  $S_j$ , i.e.  $S_i, T_j$  and  $S_j$ , where  $T_j$  is the unique extension of  $S_j$  and  $S_i$  (with phase  $2\delta$ ). Thus we can choose  $\delta$  so small that any stable object other than  $S_i, T_j$  and  $S_j$  has phase larger than  $4\delta$ .

Consider the interval  $L_0 = \{\sigma_\varepsilon\}_{\varepsilon \in (-4\delta, 0]}$ , where  $\sigma_\varepsilon = \exp(\varepsilon\pi\mathbf{i}) \cdot \sigma$ . We have

$$\begin{cases} \sigma_\varepsilon \in \operatorname{U}(\mathcal{H}), & \varepsilon \in (-\delta, 0], \\ \sigma_\varepsilon \in \operatorname{U}(\mathcal{H}_i), & \varepsilon \in (-2\delta, -\delta), \\ \sigma_\varepsilon \in \operatorname{U}(\mathcal{H}_*), & \varepsilon \in (-3\delta, -2\delta), \\ \sigma_\varepsilon \in \operatorname{U}(\mathcal{H}_{ij}), & \varepsilon \in (-4\delta, -3\delta), . \end{cases}$$

Therefore  $L_0$  is homotopy to  $L_p$ . Notice that  $L_0$  is contained in the contractible 'prism'

$$\mathbf{P} = \mathbb{C} \cdot \operatorname{U}(\mathcal{H}) \cong \mathbb{C} \cdot H^n,$$

where  $H$  is the upper half plane in (2.9.1). Similarly, the path  $\mathcal{H} \rightarrow \mathcal{H}_j \rightarrow \mathcal{H}_{ij}$  is homotopy to some interval  $L'_0 = \{\sigma'_\varepsilon\}_{\varepsilon \in (-4\delta', 0]}$  in  $\mathbf{P}$ , where  $\sigma'$  is the stability condition whose heart is  $\mathcal{H}$  satisfying

$$\begin{cases} Z'(S_k) = \exp(\frac{1}{2}\pi\mathbf{i}) & k \neq i, j, \\ Z'(S_i) = \exp(3\delta'\pi\mathbf{i}), \\ Z'(S_j) = \exp(\delta'\pi\mathbf{i}), \end{cases}$$

for some small  $\delta' > 0$ . Hence such pentagon is trivial. Same argument for the square.  $\square$

**Theorem 6.1.6.** *If  $Q$  is of Dynkin type, then  $\operatorname{Stab}(Q)$  is simply connected.*

*Proof.* By Proposition 6.1.4 and Lemma 6.1.5 we know that  $\pi_1(\operatorname{EG}(Q))$  is trivial in  $\operatorname{Stab}(Q)$ . Then the theorem follows from the surjection in Lemma 6.1.2.  $\square$

## 6.2 Simply connectedness of Calabi-Yau Dynkin case

### 6.2.1 The principal component

In this subsection, we show that  $\operatorname{EG}^\circ(\Gamma_N Q)$  induces a connected component in the stability space  $\operatorname{Stab}(\mathcal{D}(\Gamma_N Q))$ .

**Lemma 6.2.1.**  *$\operatorname{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$  is finite, for any heart  $\mathcal{H} \in \operatorname{EG}^\circ(\Gamma_N Q)$ .*

*Proof.* By (5.2.2), we can assume that  $\mathcal{H} \in \operatorname{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  without lose of generality. By Theorem 5.1.1, we have isomorphism (5.1.2) and hence  $\operatorname{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  is finite by Proposition 5.3.1.

Now we claim that, for  $\mathcal{H} \in \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ , if  $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_0)$  is finite for any  $\mathcal{H}_Q[1] \leq \mathcal{H}_0 < \mathcal{H}$ , then  $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$  is also finite.

If  $\mathcal{H} \in \text{EG}_{N-1}^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ , then  $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}) \subset \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ , which implies that  $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$  is finite. Now suppose that  $\mathcal{H} \notin \text{EG}_{N-1}^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ . Let  $\mathcal{H}$  is induced from  $\widehat{\mathcal{H}} \in \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  via  $\mathcal{I}$ , and we have  $\widehat{\mathcal{H}} \notin \text{EG}_{N-1}^\circ(Q, \mathcal{H}_Q)$  by (5.1.2). By Proposition 2.2.5, for any simple  $\widehat{S} \in \text{Sim } \widehat{\mathcal{H}}$ , there is some integer  $m$  such that  $\widehat{S} \in \mathcal{H}_Q[m]$ ; and we have  $1 \leq m \leq N-1$  by Lemma 3.2.2. Since  $\widehat{\mathcal{H}} \notin \text{EG}_{N-1}^\circ(Q, \mathcal{H}_Q)$ , there exists a simple  $\widehat{S} \in \text{Sim } \widehat{\mathcal{H}}$  such that  $\mathbf{H}_{N-1}(\widehat{S}) \neq 0$ , where  $\mathbf{H}_\bullet$  is with respect to  $\mathcal{H}_Q$ . By Proposition 2.2.5,  $\widehat{S} \in \mathcal{H}_Q[N-1]$ . Then  $S = \mathcal{I}(\widehat{S}) \in \mathcal{H}_\Gamma[N-1]$ . By Lemma 3.2.2, we have (5.2.1). By the inductive assumption, we know that  $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^b)$  and  $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^{(N-2)^b})$  is finite; hence, so is

$$\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^\sharp) = \phi_S^{-1} \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^{(N-2)^b}).$$

By Lemma 5.4.1, we have

$$\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}) \subset \left( \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^b) \cup \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}_S^\sharp) \right)$$

which implies the finiteness of  $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$ . Hence the lemma follows by induction.  $\square$

**Proposition 6.2.2.**  $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H}) = \text{EG}_3(\Gamma_N Q, \mathcal{H})$ , for any heart  $\mathcal{H} \in \text{EG}^\circ(\Gamma_N Q)$ .

*Proof.* Suppose that there exists a heart  $\mathcal{H}' \in \text{EG}_3(\Gamma_N Q, \mathcal{H}) - \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$ , we claim that there is an infinite directed path

$$\mathcal{H}_1 \xrightarrow{S_1} \mathcal{H}_2 \xrightarrow{S_2} \mathcal{H}_3 \rightarrow \dots$$

in  $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$  satisfying  $\mathcal{H}_j < \mathcal{H}'$  for any  $j \in \mathbb{N}$ .

Use induction starting from  $\mathcal{H}_1 = \mathcal{H}[1]$ . Suppose we have  $\mathcal{H}_j \in \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$  such that  $\mathcal{H}_j < \mathcal{H}'$ . If for any simple  $S \in \mathcal{H}_j$ , we have  $S \in \mathcal{H}'$ , then  $\mathcal{H}' \supset \mathcal{H}_j$  which implies  $\mathcal{P}' \supset \mathcal{P}_j$ , or  $\mathcal{H}' \leq \mathcal{H}_j$ ; this contradicts to  $\mathcal{H}_j < \mathcal{H}'$ . Thus there is a simple  $S_j \in \mathcal{H}_j$  such that  $S_j \notin \mathcal{H}'$ . Notice that  $\mathcal{H}_j < \mathcal{H}' \leq \mathcal{H}[2] \leq \mathcal{H}_j[1]$ , then by Lemma 5.4.1, we have  $\mathcal{H}_{j+1} = (\mathcal{H}_j)_{S_j}^\sharp \leq \mathcal{H}' (\leq \mathcal{H}[2])$ . Notice that  $\mathcal{H}' \notin \text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$ , therefore  $\mathcal{H}_{j+1} \neq \mathcal{H}'$ , which implies the claim.

Then we have that  $\text{EG}_3^\circ(\Gamma_N Q, \mathcal{H})$  is infinite, which contradicts to the finiteness in Lemma 6.2.1.  $\square$

Similar to Section 6.1.1, we have the following results.

**Theorem 6.2.3.** We have the formula (6.1.2). Moreover, there is a principal compo-

nent

$$\mathrm{Stab}^\circ(\Gamma_N Q) = \bigcup_{\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_N Q)} \mathrm{U}(\mathcal{H})$$

in  $\mathrm{Stab}(\mathcal{D}(\Gamma_N Q))$ , which is the connected component containing  $\mathrm{U}(\mathcal{H}_\Gamma)$ .

*Proof.* By Proposition 6.2.2, we have the finiteness condition  $(\star\star)$  in [51, Section 2], and hence [51, Proposition 2.15 and Theorem 2.17] apply which implies the theorem.  $\square$

We will also call a term in the RHS in (6.1.2) a face of the  $n$ -cell  $\mathrm{U}(\mathcal{H})$ , for any  $\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_N Q)$ . Similarly, codimension one faces of  $\mathcal{H}$  corresponds to its simple tilts and we have the corresponding canonical embedding and surjection as in Section 6.1.1.

**Proposition 6.2.4.** *There is a canonical embedding (unique up to homotopy)*

$$\iota : \mathrm{EG}^\circ(\Gamma_N Q) \hookrightarrow \mathrm{Stab}^\circ(\Gamma_N Q) \tag{6.2.1}$$

such that the conditions 1 $^\circ$  and 2 $^\circ$  in Lemma 6.1.1. Moreover, we have a surjection  $\pi_1(\mathrm{EG}^\circ(Q)) \rightarrow \pi_1(\mathrm{Stab}^\circ(Q))$ .

## 6.2.2 Simply connectedness

**Theorem 6.2.5.** *Suppose that  $Q$  is of Dynkin type. Let  $\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_N Q)$ . Then  $\pi_1(\mathrm{Stab}^\circ(\Gamma_N Q)/\mathrm{Br}, [\mathcal{H}])$  is generated by basic cycles containing  $[\mathcal{H}]$  and it is a quotient group of the braid group  $\mathrm{Br}_Q$ .*

*Proof.* Let  $\mathrm{Sim} \mathcal{H} = \{S_1, \dots, S_n\}$ ,  $\phi_k = \phi_{S_k}$  and let  $c_k$  be the basic cycle corresponding to  $l(\mathcal{H}, S_k)$ , for  $k = 1, \dots, n$ . Denote by  $p$  the quotient map

$$p : \mathrm{Stab}^\circ(\Gamma_N Q) \rightarrow \mathrm{Stab}^\circ(\Gamma_N Q)/\mathrm{Br}.$$

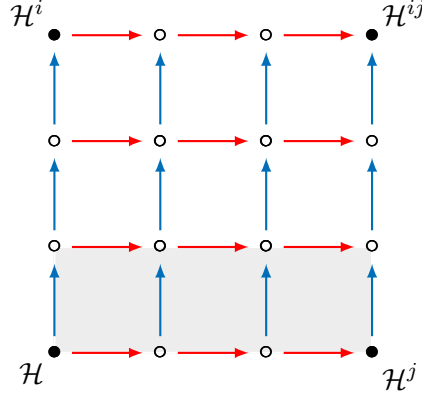
We will drop  $Y \in \{\mathrm{Stab}^\circ(\Gamma_N Q)/\mathrm{Br}, \mathrm{Stab}^\circ(\Gamma_N Q)\}$  in  $\pi_1(Y, y)$  if there is no ambiguity. By [18, Theorem 13.11], we have a short exact sequence

$$0 \longrightarrow p_*(\pi_1(\mathcal{H})) \longrightarrow \pi_1([\mathcal{H}]) \xrightarrow{\varrho} \mathrm{Br}(\Gamma_N Q) \longrightarrow 0, \tag{6.2.2}$$

where  $\varrho$  sends  $c_k$  to  $\phi_k^{-1}$ . We claim that  $\{c_k\}$  satisfies the braid group relation and generates  $\pi_1([\mathcal{H}])$ . If so, the theorem follows.

For  $i, j$  satisfying  $\mathrm{Hom}^\bullet(S_i, S_j) = 0$ , consider the lifting  $L_1$  of

$$c_i c_j c_i^{-1} c_j^{-1} \in \pi_1([\mathcal{H}])$$



**Figure 6-2:** Square cover of  $L_1$ , CY-4 case

in  $\pi_1(\mathcal{H})$  starting at  $\mathcal{H}$ . Let

$$\begin{aligned}\mathcal{H}^i &= \phi_i^{-1}(\mathcal{H}), & \mathcal{H}^{ji} &= \phi_j^{-1} \circ \phi_i^{-1}(\mathcal{H}), \\ \mathcal{H}^j &= \phi_j^{-1}(\mathcal{H}), & \mathcal{H}^{ij} &= \phi_i^{-1} \circ \phi_j^{-1}(\mathcal{H}).\end{aligned}$$

We have  $\mathcal{H}^{ij} = \mathcal{H}^{ji}$  in this case. Denote the path (of length  $N - 1$  with direction  $S$ )

$$\mathcal{H} \rightarrow \cdots \rightarrow \mathcal{H}_S^{\pm k\sharp} \rightarrow \cdots \rightarrow \mathcal{H}_S^{\pm(N-1)\sharp}$$

by  $\mathcal{H} \xrightarrow{\phi^-} \phi_S^{-1}(\mathcal{H})$  and  $\mathcal{H} \xrightarrow{\phi^+} \phi_S(\mathcal{H})$  respectively. Then  $L_1 \in \pi_1(\mathcal{H})$  is

$$\begin{array}{ccc}\mathcal{H}^i & \xrightarrow{\phi^-} & \mathcal{H}^{ij} \\ \uparrow \phi^- & & \downarrow \phi^+ \\ \mathcal{H} & \xleftarrow{\phi^+} & \mathcal{H}^j.\end{array}$$

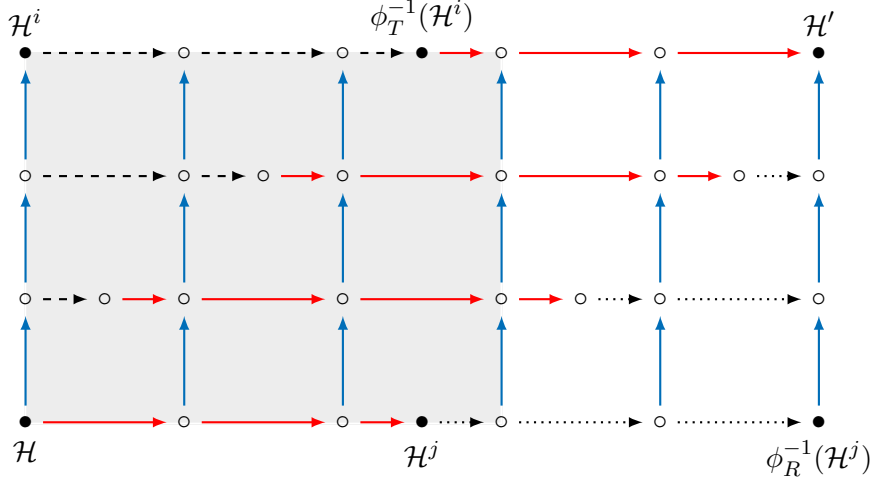
Notice that  $\dim \operatorname{Hom}^\bullet(S_j, S_i) \leq 1$  by Proposition 4.3.1. By the iterated use of Theorem 3.1.4 we can use  $(N-1)^2$  squares, as in (6.1.4), to cover  $L_1$ . For instance, Figure 6-2 is the CY-4 case, where the blue (resp. red) edges have direction  $S_i$  (resp.  $S_j$ ) and the hearts are uniquely determined by these edges. Notice that any heart in  $\operatorname{EG}^\circ(\Gamma_N Q)$  is induced hence using the same argument in Lemma 6.1.5, one can show any squares covering  $L_1$  is trivial in  $\pi_1(\mathcal{H})$ . Thus  $L_1$  is trivial in  $\pi_1(\mathcal{H})$  which implies  $c_i c_j = c_j c_i$  in  $\pi_1([\mathcal{H}])$ .

For  $i, j$  satisfying  $\operatorname{Hom}^\bullet(S_j, S_i) = \mathbf{k}[-1]$ , consider the lifting  $L_2$  of

$$c_i c_j c_i c_j^{-1} c_i^{-1} c_j^{-1} \in \pi_1([\mathcal{H}])$$

in  $\pi_1(\mathcal{H})$  that starting at  $\mathcal{H}$ . Let  $T = \phi_i^{-1}(S_j), R = \phi_j^{-1}(S_i)$ . By [47, Lemma 2.11], we

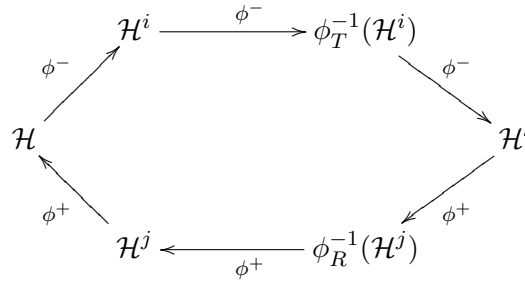



 Figure 6-3: Square and pentagon cover of  $L_2$ , CY-4 case

have

$$\phi_j^{-1} \circ \phi_T^{-1} \circ \phi_i^{-1} = \phi_i^{-1} \circ \phi_R^{-1} \circ \phi_j^{-1}.$$

Let  $\mathcal{H}' = \phi_j^{-1} \circ \phi_T^{-1} \circ \phi_i^{-1}(\mathcal{H})$ . Then  $L_2 \in \pi_1(\mathcal{H})$  is



Similarly, we can use  $(N-1)(2N-3)$  squares/pentagons, as in (6.1.4), to cover  $L_2$ . For instance, Figure 6-3 is the CY-4 case, where the blue (resp. red, dashed, dotted) edges have direction  $S_i$  (resp.  $S_j, T, R$ ). Then we deduce that  $L_2$  is trivial in  $\pi_1(\mathcal{H})$  as before. Thus  $c_i c_j c_i = c_j c_i c_j$  in  $\pi_1([\mathcal{H}])$  as required.

To finish, we only need to show that  $\{c_k\}_{k=1}^n$  generates  $\phi_1([\mathcal{H}])$ . By Theorem 5.2.1, we have  $\text{EG}(\Gamma_N Q)/\text{Br} \cong \overline{\text{EG}}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  and hence  $\pi_1(\text{EG}(\Gamma_N Q)/\text{Br})$  is generated by all squares and pentagons in  $\text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  and basic cycles. These squares and pentagons are trivial as in Lemma 6.1.5. Therefore, it is essential to show that another basic cycle that does not contain  $[\mathcal{H}]$  in  $\pi_1([\mathcal{H}])$  is generated by  $\{c_k\}_{k=1}^n$ .

Let  $\mathcal{H}_i = \mathcal{H}_{S_i}^\sharp$ ,  $T = \phi_i^{-1}(S_j)$ ,  $c_T$  be the basic cycle induced by the line  $l(\mathcal{H}_i, T)$  and  $s_i$  be the path from  $\mathcal{H}$  to  $\mathcal{H}_i$  in the line  $l(\mathcal{H}, S)$ . Consider the basic cycle  $s_i c_T s_i^{-1}$ . If  $\text{Hom}^\bullet(S_j, S_i) = 0$ , let  $L_3$  be the lifting of

$$(s_i c_T s_i^{-1}) c_i^{-1} \in \pi_1([\mathcal{H}])$$

in  $\pi_1(\mathcal{H})$  stating at  $\mathcal{H}$ . As the gray area in Figure 6-2, we can cover  $L_3$  using part of the covering for  $L_1$  which implies  $s_i c_T s_i^{-1} = c_i$ . If  $\text{Hom}^\bullet(S_j, S_i) = \mathbf{k}[-1]$ , let  $L_4$  be the lifting of

$$c_j(s_i c_T s_i^{-1}) c_j^{-1} c_i^{-1} \in \pi_1([\mathcal{H}])$$

in  $\pi_1(\mathcal{H})$  stating at  $\mathcal{H}$ . Similarly, we can cover  $L_4$  using part of the covering for  $L_2$  (as the gray area in Figure 6-3) which implies  $s_i c_T s_i^{-1} = c_j^{-1} c_i c_j$ . Either way,  $s_i c_T s_i^{-1}$  is generated by  $\{c_k\}_{k=1}^n$  as required.  $\square$

**Corollary 6.2.6.** *Let  $Q$  be a Dynkin quiver. If the braid group action on  $\mathcal{D}(\Gamma_N Q)$  is faithful, i.e.  $\text{Br}(\Gamma_Q Q) \cong \text{Br}_Q$ , then  $\text{Stab}^\circ(\Gamma_N Q)$  is simply connected. In particular, this is true for  $Q$  of type  $A_n$  or  $N = 2$ .*

*Proof.* If  $\text{Br}(\Gamma_Q Q) \cong \text{Br}_Q$ , then from (6.2.2) we deduce that  $\varrho$  is an isomorphism. Hence  $\pi_1(\text{Stab}^\circ(\Gamma_N Q)) = 1$  which implies the simply connectedness. The faithfulness for  $Q$  of type  $A_n$  follows from [47] and faithfulness for  $N = 2$  follows from [4].  $\square$

We have the following sensible conjectures.

**Conjecture 6.2.7.** *For any acyclic quiver  $Q$ ,  $\text{Br}(\Gamma_N Q) \cong \text{Br}_Q$ .*

**Conjecture 6.2.8.** *For a Dynkin quiver  $Q$ ,  $\text{Stab}(\mathcal{D}(Q))$  and  $\text{Stab}^\circ(\mathcal{D}(\Gamma_N Q))$  are contractible.*

### 6.3 A limit formula for stability spaces

We can refine the limit formula of exchange graphs in Section 5.3 to a formula of stability spaces, but only prove it for the Dynkin case.

**Lemma 6.3.1.** *If  $\mathcal{H} = F_*(\widehat{\mathcal{H}})$  for some heart  $\widehat{\mathcal{H}} \in \text{EG}^\circ(Q)$ , then a stability condition  $\widehat{\sigma} = (\widehat{Z}, \widehat{\mathcal{P}})$  on  $\mathcal{D}(Q)$  with heart  $\widehat{\mathcal{H}}$  canonically induces a stability condition  $\sigma = (Z, \mathcal{P})$  with heart  $\mathcal{H}$  and such that  $Z(F(\widehat{S})) = \widehat{Z}(\widehat{S})$  for any  $\widehat{S} \in \text{Sim } \widehat{\mathcal{H}}$ . Thus we have a homomorphism  $F_* : \text{U}(\widehat{\mathcal{H}}) \rightarrow \text{U}(\mathcal{H})$ .*

*Proof.* The heart  $\widehat{\mathcal{H}}$  and  $\mathcal{H}$  are both good by Theorem 3.1.4 and Theorem 5.2.1. Thus the lemma follows by Proposition 2.9.3.  $\square$

**Theorem 6.3.2.** *We have*

$$\text{Stab}(Q) \cong \lim_{N \rightarrow \infty} \text{Stab}^\circ(\Gamma_N Q) / \text{Br}(\Gamma_N Q)$$

*in the following sense:*

- 1°. *There exists a family of open subspaces  $\{\mathcal{S}_N\}_{N \geq 2}$  in  $\text{Stab}^\circ(Q)$  satisfying  $\mathcal{S}_N \subset \mathcal{S}_{N+1}$  and  $\text{Stab}(Q) \cong \lim_{N \rightarrow \infty} \mathcal{S}_N$ .*

$\mathcal{S}_N$  is homomorphic to a fundamental domain for  $\text{Stab}^\circ(\Gamma_N Q)/\text{Br}$ .

*Proof.* Let  $\text{Stab}_N^\circ(Q)$  and  $\text{Stab}_N^\circ(\Gamma_N Q)$  be the interior of

$$\bigcup_{\mathcal{H} \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)} \overline{U(\mathcal{H})} \quad \text{and} \quad \bigcup_{\mathcal{H} \in \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)} \overline{U(\mathcal{H})}$$

respectively. By (6.1.2), we know that a face  $F_Q$  of some cell  $U(\widehat{\mathcal{H}})$  is in  $\text{Stab}_N^\circ(Q)$  if and only if  $F_Q = \overline{U(\widehat{\mathcal{H}})} \cap U(\widehat{\mathcal{H}}')$  for some  $\widehat{\mathcal{H}}, \widehat{\mathcal{H}}' \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)$  satisfying  $\widehat{\mathcal{H}}[-1] \leq \widehat{\mathcal{H}}' < \widehat{\mathcal{H}}$ . Similarly, a face  $F_\Gamma$  of some cell  $U(\mathcal{H})$  is in  $\text{Stab}_N^\circ(Q)$  if and only if  $F_\Gamma = \overline{U(\mathcal{H})} \cap U(\mathcal{H}')$  in  $\text{Stab}_N^\circ(\Gamma_N Q)$  for some  $\mathcal{H}, \mathcal{H}' \in \text{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$  satisfying  $\mathcal{H}[-1] \leq \mathcal{H}' < \mathcal{H}$ . Notice that we have isomorphism (5.1.2), and formulae (6.1.2) for both  $\text{Stab}_N^\circ(Q)$  and  $\text{Stab}_N^\circ(\Gamma_N Q)$ . Then by Lemma 6.3.1, we know that any such face  $F_\Gamma$  in  $\text{Stab}_N^\circ(\Gamma_N Q)$  is induced from some face  $F_Q$  in  $\text{Stab}_N^\circ(Q)$  via  $\mathcal{I}$ , in the sense that we have

$$\mathcal{I}_*(F_Q) = \mathcal{I}(\overline{U(\widehat{\mathcal{H}})} \cap U(\widehat{\mathcal{H}}')) = \overline{\mathcal{I}_*(U(\widehat{\mathcal{H}}))} \cap \mathcal{I}_*(U(\widehat{\mathcal{H}}')) = \overline{U(\mathcal{H})} \cap U(\mathcal{H}') = F_\Gamma$$

Thus we can glue the homomorphisms in Lemma 6.3.1 to a homomorphism

$$\mathcal{I}_* : \text{Stab}_N^\circ(Q) \rightarrow \text{Stab}_N^\circ(\Gamma_N Q).$$

Let  $\mathcal{S}_N = \exp(-m\pi\mathbf{i}) \cdot \text{Stab}_N^\circ(Q)$ , for  $m = \lfloor -\frac{N}{2} \rfloor$ , where  $\cdot$  is the  $\mathbb{C}$ -action. Then 1° follows from the limit formula in Proposition 5.3.1 and we have  $\mathcal{S}_N \cong \text{Stab}_N^\circ(Q) \cong \text{Stab}_N^\circ(\Gamma_N Q)$ , which completes the proof.  $\square$

## 6.4 Center of the braid group

Seidel-Thomas ([47, Lemma 4.14]) showed that  $[N + nN - 2n]$  is the generator of the center of  $\text{Br}(\Gamma_N Q)$  for an  $A_n$ -type quiver  $Q$ . We will calculate such center for the Dynkin case.

Recall that we identify  $\text{Br}_Q$  with  $\text{Br}(\Gamma_N Q)$  and the Coxeter number  $h_Q$ , satisfying  $\tau^{h_Q} = [-2] \in \text{Aut } \mathcal{D}(Q)$ , is  $n+1, 2(n-1), 12, 18, 30$  for  $Q = A_n, D_n, E_6, E_7, E_8$  respectively. Moreover, define

$$\zeta_Q = \begin{cases} 1, & \text{if } Q \text{ is of type } A_n, D_{2n+1}, E_6; \\ 1/2, & \text{if } Q \text{ is of type } D_{2n}, E_7, E_8. \end{cases}$$

Provided the labeling of vertices as in (2.1.2), we know that  $z = \sigma^{\zeta_Q h_Q}$  generates of the center of the braid group  $\text{Br}(\Gamma_N Q)$  by [8], where

$$\sigma = \phi_{S_n} \circ \dots \circ \phi_{S_1}.$$

**Proposition 6.4.1.** *Let  $Q$  be a Dynkin quiver. The shift  $[(N-2)\zeta_Q h_Q + 2\zeta_Q]$  is in the center of the braid group  $\text{Br}(\Gamma_N Q)$ . Moreover, if  $\text{Br}(\Gamma_N Q) \cong \text{Br}_Q$ , then the shift  $[(N-2)\zeta_Q h_Q + 2\zeta_Q]$  generates the center.*

*Proof.* We claim that

$$\mathcal{I}_*(\Sigma_{N-1} \mathcal{H}_Q) = \sigma^{-1}(\mathcal{H}_\Gamma). \quad (6.4.1)$$

This follows by a direct calculation. We only show the case when  $Q$  is of type  $A_n$  with straight orientation for demonstration. Let  $\text{Sim } \mathcal{H}_\Gamma = \{S_i\}_{i=1}^n$  and  $S_i = \mathcal{I}(\widehat{S}_i)$ . Apply  $\sigma \circ [N-1]$  to  $\mathcal{H}_\Gamma$  with simples

$$\begin{array}{ccccccc} S_1 & \xrightarrow{d_1} & S_2 & \xrightarrow{d_2} & S_3 & \cdots & S_{n-1} \xrightarrow{d_{n-1}} S_n, \\ & \xleftarrow{N-d_1} & & \xleftarrow{N-d_1} & & & \xleftarrow{N-d_{n-1}} \end{array}$$

which is induced from the heart  $\mathcal{H}_Q$  with simples (via  $\mathcal{I}$ )

$$\widehat{S}_1 \xrightarrow{d_1} \widehat{S}_2 \xrightarrow{d_2} \widehat{S}_3 \rightarrow \cdots \widehat{S}_{n-1} \xrightarrow{d_{n-1}} \widehat{S}_n.$$

(the notation  $S \xrightarrow{d} T$  here means  $\text{Hom}^\bullet(S, T) = \mathbf{k}[-d]$ ). Then we get a heart with simples

$$\begin{array}{ccccccc} S_2[d_1] & \xrightarrow{d_1} & S_3[d_2] & \xrightarrow{d_2} & S_4[d_3] & \cdots & S_n[d_{n-1}] \xrightarrow{d_{n-1}} S_0[1], \\ & \xleftarrow{N-d_1} & & \xleftarrow{N-d_1} & & & \xleftarrow{N-d_{n-1}} \end{array} \quad (6.4.2)$$

which is induced from the heart with simples

$$\widehat{S}_2[d_1] \xrightarrow{d_1} \widehat{S}_3[d_2] \xrightarrow{d_2} \widehat{S}_4 \rightarrow \cdots \widehat{S}_n[d_{n-1}] \xrightarrow{d_{n-1}} \widehat{S}_0[1],$$

where  $\widehat{S}_0$  is determined by the filtration with factors

$$\left( \widehat{S}_n \left[ \sum_{i=1}^{n-1} (d_i - 1) \right], \dots, \widehat{S}_2[d_1 - 1], \widehat{S}_1 \right).$$

Notice that

$$\widehat{S}_{j+1}[d_j] = \tau(\widehat{S}_j[1]), \quad \widehat{S}_0 = \tau(\widehat{S}_n),$$

so we have

$$\mathcal{I}[\tau \circ [1](\mathcal{H}_Q)] = \sigma \circ [N-1](\mathcal{H}_\Gamma)$$

or  $\mathcal{I}[\Sigma_{N-1} \mathcal{H}_Q] = \sigma^{-1}(\mathcal{H}_\Gamma)$ . Then

$$\sigma^{-\zeta_Q h_Q}(\mathcal{H}_\Gamma) = \mathcal{I}_*(\Sigma_{N-1}^{\zeta_Q h_Q} \mathcal{H}_Q) = \mathcal{I}_*(\mathcal{H}_\Gamma)[m] = \mathcal{I}_*(\mathcal{H}_Q[m]) = \mathcal{H}_\Gamma[m],$$

where  $m = (N - 2)\zeta_Q h_Q + 2\zeta_Q$ , which implies the proposition.  $\square$

## 6.5 Directed paths and HN-strata

In this section, we will study the relations between directed paths in the exchange graph  $\text{EG}(Q)$ , HN-strata for  $\mathcal{H}_Q$ , slicings on  $\mathcal{D}(Q)$  and stability functions on  $\mathcal{H}_Q$ .

### 6.5.1 Directed paths

Let  $\text{EG}(Q; \mathcal{H}_1, \mathcal{H}_2)$  be the full subgraph of  $\text{EG}(Q)$  consisting of hearts  $\mathcal{H}_1 \leq \mathcal{H} \leq \mathcal{H}_2$ . Denote by  $\vec{\mathbf{P}}(\mathcal{H}_1, \mathcal{H}_2)$  the set of all directed paths from  $\mathcal{H}_1$  to  $\mathcal{H}_2$  in  $\text{EG}(Q; \mathcal{H}_1, \mathcal{H}_2)$ .

**Lemma 6.5.1.** *Suppose  $\mathcal{H}_1 \leq \mathcal{H}_2$ . Then  $\vec{\mathbf{P}}(\mathcal{H}_1, \mathcal{H}_2[1]) \neq \emptyset$  if at least one of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  is standard. In particular, we have*

$$\text{EG}(Q; \mathcal{H}[1], \mathcal{H}[N - 1]) = \text{EG}_N(Q, \mathcal{H}) = \text{EG}_N^\circ(Q, \mathcal{H}),$$

for any standard heart  $\mathcal{H} \in \text{EG}(Q)$ .

*Proof.* Without lose of generality, suppose that  $\mathcal{H}_1 = \mathcal{H}_Q[1]$  which is standard. For any simple  $S_i \in \text{Sim } \mathcal{H}_2$ ,  $S_i \in \mathcal{H}_Q[m_i]$  for some integer  $m_i$  by Proposition 2.2.5. Since  $\mathcal{H}_1 \leq \mathcal{H}_2$ , we have  $m_i \geq 1$ . Choose  $N \gg 1$  such that  $\mathcal{H}_2 \in \text{EG}_N^\circ(Q, \mathcal{H}_Q)$  and then  $\#\text{Ind}(\mathcal{P}_1 - \mathcal{P}_2)$  is finite. If  $\mathcal{H}_1 < \mathcal{H}_2$ , there exists  $j$  such that  $m_j > 1$ . By Lemma 3.2.2, we can backward tilt  $\mathcal{H}_2$  to  $(\mathcal{H}_2)_{S_j}^\flat$  within  $\text{EG}_N^\circ(Q, \mathcal{H}_Q)$  which reduces  $\#\text{Ind}(\mathcal{P}_1 - \mathcal{P}_2)$ . Thus we can iterated backward tilt  $\mathcal{H}_2$  to  $\mathcal{H}_1$  by induction, which implies the lemma.  $\square$

Define the directed distance  $\text{dis}(\mathcal{H}_1, \mathcal{H}_2)$  and diameter  $\text{diam}(\mathcal{H}_1, \mathcal{H}_2)$  between  $\mathcal{H}_1$  and  $\mathcal{H}_2$  to be the minimum and respectively maximum over the lengths of the paths in  $\vec{\mathbf{P}}(\mathcal{H}_1, \mathcal{H}_2)$ . Recall that  $\text{pf}$  is the position function defined in Definition/Lemma 4.1.5 and  $h_Q$  is the Coxeter number (see Section 6.4). Since  $\tau^{h_Q} = [-2]$ , we have

$$\text{pf}(M[1]) - \text{pf}(M) = h_Q, \quad \forall M \in \Lambda(\mathcal{D}(Q)).$$

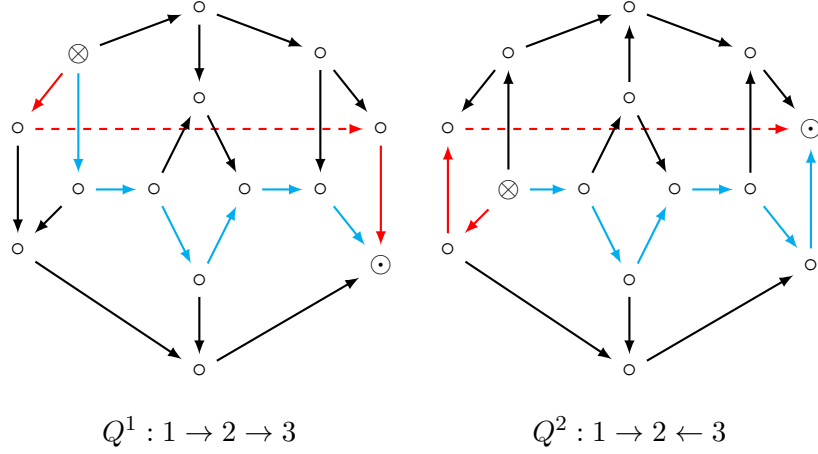
There are the following easy estimations.

**Lemma 6.5.2.** *Suppose that  $\vec{\mathbf{P}}(\mathcal{H}_1, \mathcal{H}_2) \neq \emptyset$ . Let  $\mathcal{P}_i$  be the  $t$ -structure corresponding to  $\mathcal{H}_i$ . We have*

$$\text{diam}(\mathcal{H}_1, \mathcal{H}_2) \leq \#\text{Ind}(\mathcal{P}_1 - \mathcal{P}_2) \tag{6.5.1}$$

$$\text{diam}(\mathcal{H}_1, \mathcal{H}_2) \leq \#\text{Ind}(\mathcal{P}_2^\perp - \mathcal{P}_1^\perp) \tag{6.5.2}$$

$$\text{dis}(\mathcal{H}_1, \mathcal{H}_2) \geq \frac{\text{pf}(\mathcal{H}_2) - \text{pf}(\mathcal{H}_1)}{h_Q}. \tag{6.5.3}$$



**Figure 6-4:** longest and shortest paths from  $\mathcal{H}$  to  $\mathcal{H}[1]$

In particular  $\text{dis}(\mathcal{H}, \mathcal{H}[m]) \geq nm$  and equality holds if  $\mathcal{H}$  is standard.

*Proof.* For any edge  $\mathcal{H} \rightarrow \mathcal{H}_S^\sharp$ , we have  $\text{Ind } \mathcal{P} \supsetneq \text{Ind } \mathcal{P}_S^\sharp$  by Lemma 2.3.4 and hence (6.5.1) follows. Similarly for (6.5.2).

By Theorem 3.1.4 we have formula (3.1.6). Notice that  $T_j$  is a successor of  $S_j$  and hence  $\text{pf}(T_j) > \text{pf}(S_j)$ . We have

$$\text{pf}(\mathcal{H}_S^\sharp) - \text{pf}(\mathcal{H}) = \text{pf}(S[1]) - \text{pf}(S) + \sum_{j \in J_i^\sharp} (\text{pf}(T_j) - \text{pf}(S_j)) \geq \text{pf}(S[1]) - \text{pf}(S) = h_Q$$

which implies the inequality (6.5.3). In particular, if  $\mathcal{H}_1 = \mathcal{H}, \mathcal{H}_2 = \mathcal{H}[m]$ , the RHS of (6.5.3) equals  $mn$ .

Now suppose  $\mathcal{H}$  is standard, without loss of generality let  $\mathcal{H} = \mathcal{H}_Q$ . Label the simples  $S_1, \dots, S_n$  such that  $\text{pf}(S_1) \leq \text{pf}(S_2) \leq \dots \leq \text{pf}(S_n)$ . By Lemma 4.1.3,  $\text{Hom}(M, L) \neq 0$  implies  $L$  is a successor of  $M$  and hence  $\text{pf}(M) < \text{pf}(L)$ . Thus  $\text{Hom}^1(S_i, S_j) = 0$  for  $i > j$ . By Theorem 3.1.4 we can tilt from  $\mathcal{H}$  to  $\mathcal{H}[1]$  with respect to the simples  $S_n, \dots, S_1$  in order, which implies  $\text{dis}(\mathcal{H}, \mathcal{H}[m]) = mn$ .  $\square$

**Example 6.5.3.** Let  $Q^1$  and  $Q^2$  be the quivers in Figure 6-4 while  $\mathcal{H}_{Q^i}$  and  $\mathcal{H}_{Q^i}[1]$  are  $\otimes$  (the source) and  $\odot$  (the sink), respectively, in the figure. Then we have

$$\text{diam}(\mathcal{H}_{Q^i}, \mathcal{H}_{Q^i}[1]) = 6, \quad \text{dis}(\mathcal{H}_{Q^i}, \mathcal{H}_{Q^i}[1]) = 3. \quad (6.5.4)$$

where the blue and red paths are the longest and shortest respectively.

We can give a characterization of the longest paths in  $\vec{\mathbf{P}}(\mathcal{H}_Q, \mathcal{H}_Q[1])$ .

**Proposition 6.5.4.** *Let  $\mathcal{H}$  be a standard heart, then we have*

$$\text{diam}(\mathcal{H}, \mathcal{H}[1]) = \# \text{Ind } \mathcal{H}_Q = |Q_0| \cdot h_Q / 2. \quad (6.5.5)$$

Moreover, a path  $p$  in  $\vec{\mathbf{P}}(\mathcal{H}, \mathcal{H}[1])$  has the longest length if and only if all vertices of  $p$  are standard hearts.

*Proof.* We can tilt from  $\mathcal{H}$  to  $\mathcal{H}[1]$  by a sequence of APR-tiltings, which are L-tiltings. By Corollary 4.2.2, such a path consisting of L-tiltings has length

$$\# \text{Ind}(\mathcal{P} - \mathcal{P}[1]) = \# \text{Ind}(\mathcal{P}[1]^\perp - \mathcal{P}^\perp) = \# \text{Ind } \mathcal{H}_Q.$$

Then the first claim follows from (6.5.1).

Suppose  $p$  is a longest path and use induction starting from  $\mathcal{H}_Q$  which is standard. Consider an edge  $\mathcal{H} \rightarrow \mathcal{H}_S^\sharp$  in  $p$  with  $\mathcal{H}$  is standard. Since  $p$  is longest, by (6.5.1), we have

$$\# \text{Ind}(\mathcal{P} - \mathcal{P}_S^\sharp) = 1.$$

Notice that  $S \in (\mathcal{P} - \mathcal{P}_S^\sharp)$ , we have

$$\text{Ind } \mathcal{P}_S^\sharp = \text{Ind } \mathcal{P} - \{S\}.$$

Similarly, we have

$$\text{Ind} \left( \mathcal{P}_S^\sharp \right)^\perp = \text{Ind}(\mathcal{P})^\perp \cup \{S\}.$$

and hence

$$\text{Ind } \mathcal{P} \cup \text{Ind } \mathcal{P}^\perp = \text{Ind } \mathcal{P}_S^\sharp \cup \text{Ind} \left( \mathcal{P}_S^\sharp \right)^\perp. \quad (6.5.6)$$

By Proposition 4.1.4, the fact that a heart  $\mathcal{H}'$  is standard is equivalent to

$$\text{Ind } \mathcal{D}(Q) = \text{Ind } \mathcal{P}' \cup \text{Ind}(\mathcal{P}')^\perp.$$

Therefore, by (6.5.6), the standardness of  $\mathcal{H}$  implies the standardness of  $\mathcal{H}_S^\sharp$ . Thus the necessity follows.

On the other hand, if  $\mathcal{H}$  and its simple forward tilts  $\mathcal{H}_S^\sharp$  are standard, we claim that it is an APR-tilting at a sink. Suppose not, that the vertex  $V \in Q_0$  corresponding to  $S$  is not a sink. Then there is an edge  $(V \rightarrow V') \in Q_1$  which corresponds to a nonzero map in  $\text{Ext}^1(S, S')$ , where  $S'$  is the simple corresponding to  $V'$ . Then  $S \notin (\mathcal{P}_S^\sharp)^\perp$  since  $S'[1] \in \mathcal{P}[1] \subset \mathcal{P}_S^\sharp$  by Lemma 2.3.4. Notice that  $S \notin \mathcal{P}_S^\sharp$ , we know that  $\mathcal{H}_S^\sharp$  is not standard by Proposition 4.1.4, which is a contradiction. Thus if all the vertices of a path  $p$  are standard then it consisting of APR-tiltings, which are L-tiltings. By Corollary 4.2.2, we know that the length of  $p$  is  $\# \text{Ind } \mathcal{H}_Q$  which implies  $p$  is longest.  $\square$

### 6.5.2 HN-strata

In this subsection, we use Reineke's notion of HN-strata to give an algebraic interpretation of

$$\vec{\mathbf{P}}(Q) := \vec{\mathbf{P}}(\mathcal{H}_Q, \mathcal{H}_Q[1]).$$

**Definition 6.5.5.** A (discrete) *HN-stratum*  $[T_l, \dots, T_1]_{\text{HN}}$  in an abelian category  $\mathcal{C}$  is an ordered collection of objects  $T_l, \dots, T_1$  in  $\text{Ind } \mathcal{C}$ , satisfying the HN-property:

- $\text{Hom}(T_i, T_j) = 0$  for  $i > j$ .
- For any nonzero object  $M$  in  $\mathcal{C}$ , there is an HN-filtration by short exact sequences

$$\begin{array}{ccccccc}
 0 = M_0 & \longrightarrow & M_1 & \longrightarrow & \dots & \longrightarrow & M_{m-1} & \longrightarrow & M_m = M \\
 & & \searrow & & & & \nwarrow & & \\
 & & A_{j_1} & & & & A_{j_m} & & 
 \end{array} \quad (6.5.7)$$

with  $A_{j_i}$  is in  $\langle T_{j_i} \rangle$  and  $1 \leq j_1 < \dots < j_m \leq l$ .

Notice that the uniqueness of HN-filtration follows from the first condition in HN-property. Denote by  $\text{HN}(Q)$  the set of all HN-strata of  $\mathcal{H}_Q$ . We claim that there is a bijection between  $\vec{\mathbf{P}}(Q)$  and  $\text{HN}(Q)$ .

Let  $p = T_l \cdot \dots \cdot T_1$  be a path in  $\vec{\mathbf{P}}(Q)$

$$p : \mathcal{H}_Q = \mathcal{H}_0 \xrightarrow{T_1} \mathcal{H}_1 \xrightarrow{T_2} \dots \xrightarrow{T_l} \mathcal{H}_l = \mathcal{H}_Q[1]$$

with corresponding t-structures  $\mathcal{P}_0 \supset \mathcal{P}_1 \supset \dots \supset \mathcal{P}_l$ . We have the following lemmas.

**Lemma 6.5.6.** *For any indecomposable  $M$  in  $\mathcal{H}_Q$ , there is a filtration as (6.5.7) such that  $A_{j_i}$  is in  $\langle T_{j_i} \rangle$  and  $1 \leq j_1 < \dots < j_m \leq l$ .*

*Proof.* We construct such a filtration as follows. Since

$$M \in \mathcal{P}_0 - \mathcal{P}_l = \bigcup_{i=1}^l (\mathcal{P}_{i-1} - \mathcal{P}_i),$$

there exists an integer  $0 < j \leq l$  such that  $M \in \mathcal{P}_{j-1} - \mathcal{P}_j$ . Since  $\mathcal{H}_j = (\mathcal{H}_{j-1})_{T_j}^\sharp$ , we have a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow A_j \longrightarrow 0$$

such that  $A_j$  in  $\langle T_j \rangle$ . This is the last short exact sequence in the required filtration.



Since  $M'$  is in the torsion part corresponding to  $(\mathcal{H}_{j-1})_{T_j}^\sharp$ , we have

$$M' \in \mathcal{P}_j - \mathcal{P}_l = \bigcup_{i=j}^l (\mathcal{P}_{i-1} - \mathcal{P}_i).$$

Therefore we can repeat the procedure above and the lemma follows by induction.  $\square$

**Lemma 6.5.7.** *Let  $0 \leq j \leq l$ . Let  $\mathcal{F}_j = \langle T_1, \dots, T_j \rangle$  and  $\mathcal{T}_j = \langle T_{j+1}, \dots, T_l \rangle$ . Then  $(\mathcal{F}_j, \mathcal{T}_j)$  is a torsion pair in  $\mathcal{H}_Q$  and  $\mathcal{H}_j = (\mathcal{H}_Q)^\sharp$  with respect to this torsion pair.*

*Proof.* Use induction on  $j$  starting from the trivial case when  $j = 0$ . Now suppose that  $\mathcal{H}_j = (\mathcal{H}_Q)^\sharp$  with respect to  $(\mathcal{F}_j, \mathcal{T}_j)$ . Since  $T_{j+1}$  is a simple in  $\mathcal{H}_{j+1}$  and  $T_k \in \mathcal{P}_{j+1}$  for  $k > j + 1$ , we have  $\text{Hom}(T_k, T_{j+1}) = 0$  which implies  $\text{Hom}(A, B) = 0$  for  $A \in \mathcal{T}_{j+1}, B \in \mathcal{F}_{j+1}$ . By Lemma 6.5.6 we know that for any object  $M$  in  $\text{Ind } \mathcal{H}_Q$ , there is a short exact sequence  $0 \rightarrow A \rightarrow M \rightarrow B \rightarrow 0$  such that  $A \in \mathcal{T}_{j+1}$  and  $B \in \mathcal{F}_{j+1}$ . Therefore  $(\mathcal{F}_{j+1}, \mathcal{T}_{j+1})$  is a torsion pair in  $\mathcal{H}_Q$ . By Proposition 2.3.5, we have  $\mathcal{H}_j \cap \mathcal{H}_Q = \mathcal{T}_j$ . To finish we only need to show that  $\mathcal{H}_{j+1} \cap \mathcal{H}_Q = \mathcal{T}_{j+1}$ . This follows from  $\mathcal{H}_{j+1} = (\mathcal{H}_j)_{T_j}^\sharp$ .  $\square$

Now we have an injection  $\vec{\mathbf{P}}(Q) \rightarrow \text{HN}(Q)$  as follows.

**Corollary 6.5.8.** *Any directed path  $p = p = T_l \cdot \dots \cdot T_1$  in  $\vec{\mathbf{P}}(Q)$  induces an HN-stratum  $[T_l, \dots, T_1]_{\text{HN}}$  in  $\text{HN}(Q)$ .*

*Proof.* Since  $T_i \in \mathcal{F}_j$  and  $T_j \in \mathcal{T}_j$  for  $j > i$ ,  $\text{Hom}(T_j, T_i) = 0$  follows from Lemma 6.5.7. Together with Lemma 6.5.6, the corollary follows.  $\square$

For the converse construction, we have the following lemma.

**Lemma 6.5.9.** *Let  $[T_l, \dots, T_1]_{\text{HN}}$  be an HN-stratum. For  $0 \leq j \leq l$ , let  $\mathcal{F}_j = \langle T_1, \dots, T_j \rangle$  and  $\mathcal{T}_j = \langle T_{j+1}, \dots, T_l \rangle$ . Then  $(\mathcal{F}_j, \mathcal{T}_j)$  is a torsion pair in  $\mathcal{H}_Q$ . Let  $\mathcal{H}_j = (\mathcal{H}_Q)^\sharp$  with respect to this torsion pair. Then  $T_{j+1}$  is a simple in  $\mathcal{H}_j$  and  $\mathcal{H}_{j+1} = (\mathcal{H}_j)_{T_{j+1}}^\sharp$ .*

*Proof.* Similar to Lemma 6.5.7.  $\square$

Combine the lemmas above, we have the following theorem.

**Theorem 6.5.10.** *The HN-stratas in  $\text{HN}(Q)$  are precisely the directed paths in  $\vec{\mathbf{P}}(Q)$ .*

We will not distinguish  $\text{HN}(Q)$  and  $\vec{\mathbf{P}}(Q)$  from now on.

**Corollary 6.5.11.** *For any shortest path  $p$  in  $\vec{\mathbf{P}}(Q)$ , the set of labels of its edges are precisely  $\text{Sim } \mathcal{H}_Q$ .*

*Proof.* The HN-filtration of a simple in  $\mathcal{H}_Q$  (with respect to  $p$ ) can only have one factor, i.e. itself. Hence any simple of  $\mathcal{H}_Q$  appears in an HN-stratum, and in particular, the labels of edges of  $p$ . Thus the length of  $p$  is at least  $n$ . By Lemma 6.5.2, the length of a shortest path  $p$  is exactly  $n$  and hence the corollary follows.  $\square$

### 6.5.3 Slicing interpretation

We say a slicing  $\mathcal{S}$  of  $\mathcal{D}(Q)$  is discrete if the abelian category  $\mathcal{S}(\phi)$  is either zero or contains exactly one simple for any  $\phi \in \mathbb{R}$ . We say a heart  $\mathcal{H}$  is in a slicing  $\mathcal{S}$  if  $\mathcal{H} = \mathcal{S}[\phi, \phi + 1)$  or  $\mathcal{H} = \mathcal{S}(\phi, \phi + 1]$  for some  $\phi \in \mathbb{R}$ . Let  $\text{Sli}^*(\mathcal{D}(Q), \mathcal{H})$  be the set of all discrete slicings of  $\mathcal{D}(Q)$  that contain  $\mathcal{H}$ .

**Definition 6.5.12.** Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  in  $\text{Sli}(\mathcal{D})$ . If there is a monotonic function  $\mathbb{R} \rightarrow \mathbb{R}$  such that  $\mathcal{S}_1(\phi) = \mathcal{S}_2(f(\phi))$ , then we say that the slicing  $\mathcal{S}_1$  is homotopic ( $\sim$ ) to  $\mathcal{S}_2$ .

Now we can describe the relation between directed paths and slicings.

**Proposition 6.5.13.** *There is a canonical bijection  $\text{Sli}^*(\mathcal{D}(Q), \mathcal{H}_Q)/\sim \rightarrow \text{HN}(Q)$ .*

*Proof.* Let  $\mathcal{S} \in \text{Sli}^*(\mathcal{D}(Q), \mathcal{H}_Q)$  and suppose  $\mathcal{H}_Q = \mathcal{S}(I)$  for some interval  $I$  with  $|I| = 1$ . Then it induces an HN-stratum by taking the collection of objects which are simple in  $\mathcal{S}(\phi)$  for  $\phi \in I$  with decreasing order. On the other hand, an HN stratum  $[K_l, \dots, K_1]_{\text{HN}}$  is induced by the slicing

$$\{\mathcal{P}(m + \frac{j}{l}) = \langle K_j[m] \rangle \mid j = 1, \dots, l, m \in \mathbb{Z}, \}.$$

Hence we have a surjection  $\text{Sli}^*(\mathcal{D}(Q), \mathcal{H}_Q) \rightarrow \text{HN}(Q)$  while the condition that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  maps to one HN-stratum is exactly the homotopy condition.  $\square$

### 6.5.4 Total stability

Recall that we have the notion of a stability function on an abelian category (Definition 2.9.2). We call a stability function on  $\mathcal{A}$  *totally stable* if every indecomposable is stable. Reineke made the following conjecture.

**Conjecture 6.5.14** ([49]). *Let  $Q$  be a Dynkin quiver. There exists a totally stable stability function on  $\mathcal{H}_Q$*

This was first proved by Hille-Juteau (unpublished, see the comments after [32, Corollary 1.7]).

We say a stability condition on a triangulated category is *totally stable* if any indecomposable is stable. Let  $\sigma = (Z, \mathcal{P})$  be a totally stable stability condition. Then it will induce a totally stable stability function  $Z$  on any abelian category  $\mathcal{P}(I)$ , for any half open half closed interval  $I \subset \mathbb{R}$  with length 1; in particular, on its heart. On the other hand, a totally stable stability function on  $\mathcal{H}_Q$  will induce a stability condition on  $\mathcal{D}(Q)$ , which is also totally stable.

Now we give explicit examples to prove the existence of the totally stable stability condition on  $\mathcal{D}(Q)$ , which is a slightly weak version of Conjecture 6.5.14 because orientation matters [35].

**Proposition 6.5.15.** *Let  $Q$  be a Dynkin quiver. There exists a totally stable stability condition on  $\mathcal{D}(Q)$ .*

*Proof.* We treat the cases  $A, D$  and  $E$  separately.

For  $A_n$ -type, we use [48, Example A, Section 2]. Choose the orientation of  $Q$  as

$$n \longrightarrow n-1 \longrightarrow \cdots \longrightarrow 1$$

Let the stability function  $Z$  on  $\mathcal{H}_Q$  be defined by  $Z(S_j) = -j + \mathbf{i}$ . Then  $Z$  induces a totally stable stability condition on  $\mathcal{D}(Q)$ .

For  $D_n$ -type, choose the orientation of  $Q$  as

$$n-2 \longrightarrow n-3 \longrightarrow \cdots \longrightarrow 1 \begin{array}{l} \nwarrow n-1 \\ \swarrow n \end{array}$$

Let the stability function  $Z$  on  $\mathcal{H}_Q$  be defined by

$$\begin{cases} Z(S_1) = \frac{n-3n}{2} + \mathbf{i}, \\ Z(S_j) = -j + \mathbf{i}, \quad j = 2, \dots, n-2, \\ Z(S_{n-1}) = Z(S_n) = \frac{6+3n-n^2}{4} + \mathbf{i}. \end{cases}$$

Notice that the  $\tau$ -orbit of  $S_{n-2}$  in  $\Lambda(\mathcal{H}_Q)$  is

$$P_{n-2} - \cdots - M - \cdots - S_2 - \cdots - S_3 - \cdots - S_{n-2}$$

with central charges

$$\mathbf{i}, -1 + \mathbf{i}, -2 + \mathbf{i}, -3 + \mathbf{i}, \dots, -(n-2) + \mathbf{i}$$

it is easy to check that  $Z$  induces a totally stable stability condition on  $\mathcal{D}(Q)$ .

For the exceptional case, we use Keller's quiver mutation program [36] to produce explicit examples of totally stable stability conditions for  $E_{6,7,8}$ . Choose certain orientation of  $E_6$  such that corresponding AR-quiver  $\Lambda(\mathcal{H}_Q)$  is as in Figure 6-5, Then we have a totally stable stability function such that  $\Lambda(\mathcal{H}_Q)$  is as in Figure 6-6, where the blue circle is the origin and the pink circles are the simples. Similarly we have the totally stable stability functions for  $E_7$  and  $E_8$  in Figure 6-7 and Figure 6-8 respectively.  $\square$

### 6.5.5 Inducing directed paths

We call a stability function *discrete*, if  $\mu_Z$  is injective when restricted to the stable indecomposables.

**Proposition 6.5.16.** [41] *Let  $Z : \mathcal{K}(\mathcal{H}_Q) \rightarrow \mathbb{C}$  be a discrete stability function. Then*

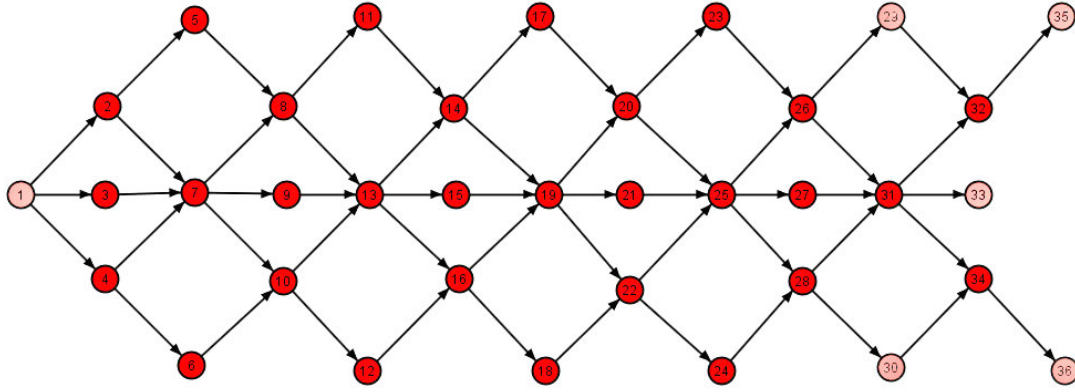


Figure 6-5: AR-quiver  $\Lambda(\mathcal{H}_Q)$  of  $E_6$

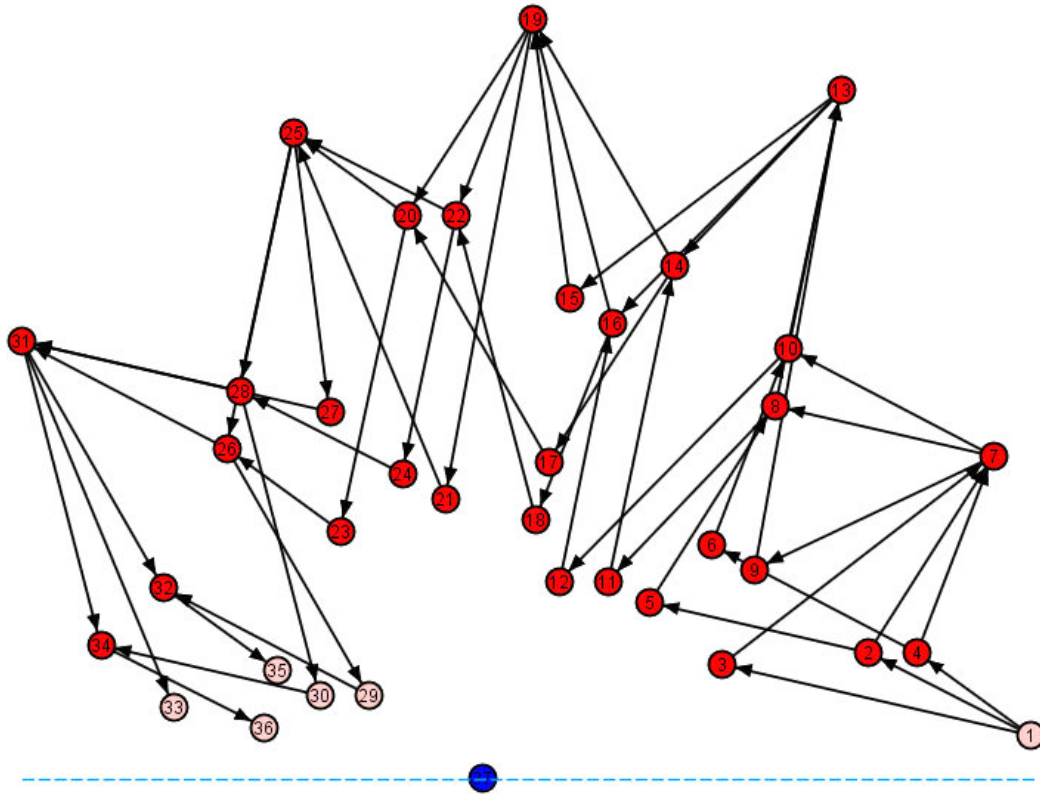
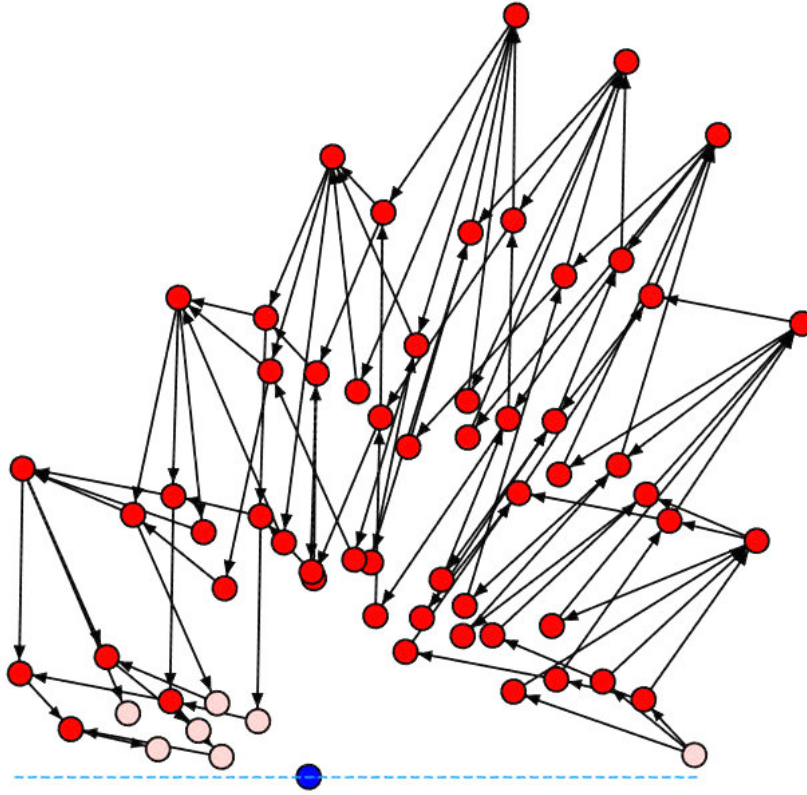
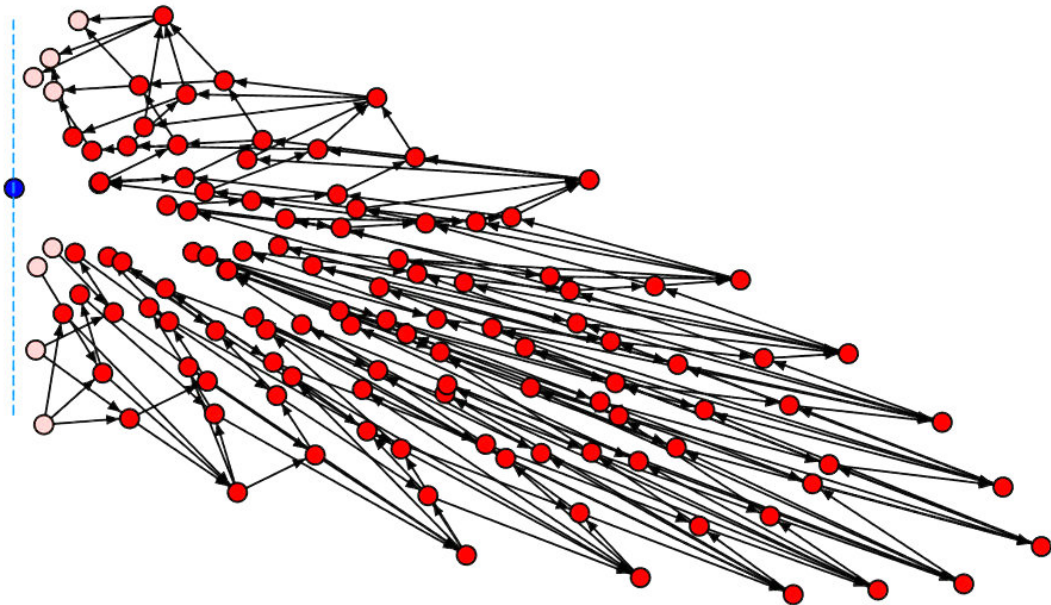


Figure 6-6: The AR-quiver  $\Lambda(\mathcal{H}_Q)$  of  $E_6$ -type under a totally stable stability function



**Figure 6-7:** *The AR-quiver of  $\Lambda(\mathcal{H}_Q)$   $E_7$ -type under a totally stable stability function*



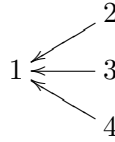
**Figure 6-8:** *The AR-quiver  $\Lambda(\mathcal{H}_Q)$  of  $E_8$ -type under a totally stable stability function*

the collection of its stable indecomposables in the order of decreasing phase is an HN-stratum of  $\mathcal{H}_Q$ .

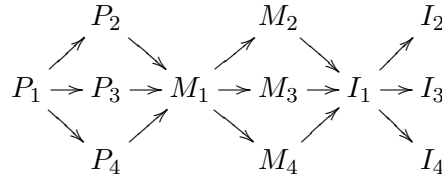
We say that a directed path in  $\vec{\mathbf{P}}(Q)$  is *induced* if the corresponding HN-stratum is induced by some discrete stability function on  $\mathcal{H}_Q$ . Notice that, a totally stable stability function on  $\mathcal{H}_Q$  induced a directed path  $p_s$  in  $\vec{\mathbf{P}}(Q)$  such that there is an edge  $M$  in  $p_s$  for any  $M \in \text{Ind } \mathcal{H}_Q$ . By (6.5.5), we know that  $p_s$  is the longest path in  $\vec{\mathbf{P}}(Q)$ . Thus, in the language of exchange graphs, Reineke's conjecture translates to, that there exists a longest path in  $\vec{\mathbf{P}}(Q)$  which is induced.

It is natural to make a very strong generalization of Reineke's conjecture, that any path in  $\vec{\mathbf{P}}(Q)$  is induced. However, this is not true, even for some longest path as below.

**Counterexample 6.5.17.** Let  $Q$  be the following quiver of  $D_4$ -type



Then the AR-quiver of  $\mathcal{H}_Q$  is

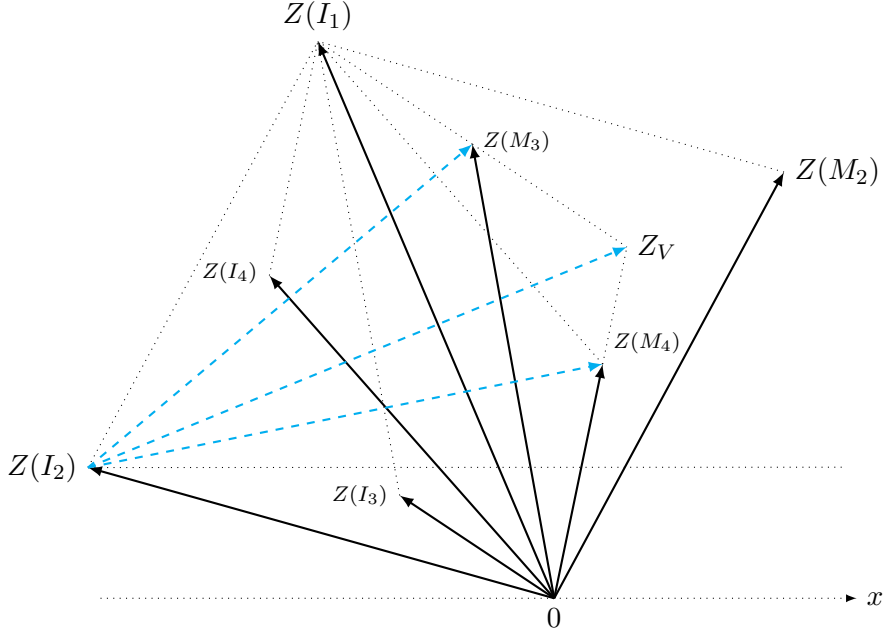


We claim that the following longest path

$$p = I_2 \cdot I_3 \cdot I_4 \cdot I_1 \cdot M_3 \cdot M_4 \cdot M_2 \cdot M_1 \cdot P_2 \cdot P_3 \cdot P_4 \cdot P_1 \quad (6.5.8)$$

is not induced. Suppose not, that  $p$  is induced by some stability function  $Z$ . The phase function  $\mu_Z$  is decreasing on the edges in  $p$  from left to right in (6.5.8). Then  $Z(I_3), Z(I_4), Z(M_3), Z(M_4)$  are in the parallelogram  $\mathfrak{P}$  with vertices  $Z(I_2), Z(I_1), Z(M_2)$  and 0, as shown in Figure 6-9. Let  $Z_V$  be the intersection of the line passing through points  $Z(I_1), Z(M_3)$  and the line passing through points  $Z(M_4), 0$ . Notice that

$$\mu_Z(P_3), \mu_Z(P_4) \in [0, \mu_Z(I_2)),$$



**Figure 6-9:** The parallelogram  $\mathfrak{P}$

we have

$$\begin{aligned}
 \mu_Z(P_3)\pi &= \arg(Z(M_4) - Z(I_2)) \\
 &< \arg(Z(M_4) - Z_V) \\
 &< \arg(Z(M_3) - Z(I_2)) \\
 &= \mu_Z(P_4)\pi,
 \end{aligned}$$

which is a contradiction.

This suggests another generalization of Reineke's conjecture as follows. We say two directed paths in  $\vec{\mathbf{P}}(Q)$  are *weakly equivalent* if the unordered sets of their edges coincide.

**Conjecture 6.5.18.** *There is an induced path in each weak equivalence class in  $\vec{\mathbf{P}}(Q)$ .*

Notice that by (6.5.5), all longest paths in  $\vec{\mathbf{P}}(Q)$  form a weak equivalent class  $E$ . Thus Reineke's conjecture can be stated as: there is a path in the weak equivalence class  $E$  which is induced.

## 6.6 Quantum dilogarithm via exchange graph

In this section, we define a DT-function on paths in exchange graphs, which provides another proof of Reineke's identities (see Theorem 6.6.1) and the existence of DT-type invariants for a Dynkin quiver.

### 6.6.1 DT-invariant for a Dynkin quiver

Let  $q$  be an indeterminate and  $\mathbb{A}_Q$  be the quantum affine space

$$\mathbb{Q}(q^{1/2})\{y^\alpha \mid \alpha \in \mathbb{N}^{Q_0}, y^\alpha y^\beta = q^{\frac{1}{2}(\langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle)} y^{\alpha+\beta}\}, \quad (6.6.1)$$

where  $\langle -, - \rangle$  is the Euler form associated to  $Q$  (see Section 2.1). Denote  $y^{\dim M}$  by  $y^M$  for  $M \in \mathcal{H}_Q$ . Notice that  $\mathbb{A}_Q$  can be also written as

$$\mathbb{Q}(q^{1/2})\langle y^S \mid S \in \text{Sim } \mathcal{H}_Q \rangle / (y^{S_i} y^{S_j} - q^{\lambda_Q(i,j)} y^{S_j} y^{S_i}),$$

where

$$\lambda_Q(i, j) = \langle S_j, S_i \rangle - \langle S_i, S_j \rangle. \quad (6.6.2)$$

Let  $\widehat{\mathbb{A}}_Q$  be the completion of  $\mathbb{A}_Q$  with respect to the ideal generated by  $y^S, S \in \text{Sim } \mathcal{H}_Q$ .

The DT-invariant  $\text{DT}(Q)$  of the quiver  $Q$  can be calculated by the product (6.6.3) as follows.

**Theorem 6.6.1** (Reineke [48], c.f. [32]). *For any HN-stratum  $\varsigma = [K_1, \dots, K_l]_{\text{HN}}$  in  $\text{HN}(Q)$ , the product*

$$\text{DT}(Q; \varsigma) = \prod_{j=1}^l \mathbb{E}(y^{K_j}) \quad (6.6.3)$$

in  $\mathbb{A}_Q$  is actually independent of  $\varsigma$ , where  $\mathbb{E}(y)$  is the quantum dilogarithm defined as the formal series

$$\mathbb{E}(y) = \sum_{j=0}^{\infty} \frac{q^{j^2/2} y^j}{\prod_{k=0}^{j-1} (q^j - q^k)}.$$

In this subsection, we will review Reineke's approach to Theorem 6.6.1, via identities in the Hall algebra (closely following [32]).

Let  $\mathbf{k}_0$  be a finite field with  $q_0 = |\mathbf{k}_0|$  and consider  $\mathcal{H}_Q(\mathbf{k}_0) = \text{mod } \mathbf{k}_0 Q$ . Recall that the completed (non twisted, opposite) Hall algebra  $\widehat{\mathbf{H}}_{\mathbf{k}_0}(Q)$  consists of formal series with rational coefficients

$$\sum_{[M] \in \mathcal{H}_Q} a_m[M],$$

where the sum is over all isomorphism classes  $[M]$  in  $\mathcal{H}_Q$ . The product in  $\widehat{\mathbf{H}}_{\mathbf{k}_0}(Q)$  is given by the formula

$$[L][M] = \sum c_{LM}^K(q_0)[K]$$

where  $c_{LM}^N(q_0)$  is the number of submodules  $L'$  of  $K$  such that  $L' \cong L$  and  $K/L' \cong M$  in  $\mathcal{H}_Q(\mathbf{k}_0)$ . Then the HN-property of an HN-stratum  $\varsigma = [K_1, \dots, K_l]_{\text{HN}}$  translates into



the identity (in Hall algebra) as

$$\sum_{[M] \in \mathcal{H}_Q} [M] = \prod_{j=1}^l \sum_{[M] \in \langle K_j \rangle} [M] \quad (6.6.4)$$

Reineke showed that there is an algebra homomorphism (called *integration*)

$$\begin{aligned} \int : \widehat{\mathbf{H}}_{\mathbf{k}_0}(Q) &\rightarrow \widehat{\mathbb{A}}_{Q, q=q_0} \\ [M] &\mapsto q^{\langle \dim M, \dim M \rangle} \frac{y^M}{|\mathrm{Aut} M|}. \end{aligned}$$

By integrating (6.6.4), a term  $\sum_{[M] \in \langle K_j \rangle} [M]$  in the RHS gives  $\mathbb{E}(y^{K_j})$  and hence the RHS gives  $\mathrm{DT}(Q; \varsigma)$ . Notice that the LHS of (6.6.4) is clearly independent of  $\varsigma$ , thus its integration gives the DT-invariant  $\mathrm{DT}(Q)$  for a Dynkin quiver  $Q$ .

**Example 6.6.2.** [32, Corollary 2.7] By the proof of Lemma 6.5.2, we know that  $\overrightarrow{\prod}_{S \in \mathrm{Sim} \mathcal{H}} S$  is a shortest path in  $\overrightarrow{\mathbf{P}}(Q)$ , where the product is with respect to the increasing order of the position function (if two objects have the same position function, then their order does not matter). Moreover, by direct checking, we know that  $\overleftarrow{\prod}_{M \in \mathrm{Ind} \mathcal{H}} M$  is a longest path in  $\overrightarrow{\mathbf{P}}(Q)$  consisting of APR tiltings, where the product is with respect to the decreasing order of the position function. Then these two paths (or the corresponding HN-strata) give the equality

$$\overleftarrow{\prod}_{M \in \mathrm{Ind} \mathcal{H}} \mathbb{E}(y^M) = \overrightarrow{\prod}_{S \in \mathrm{Sim} \mathcal{H}} \mathbb{E}(y^S). \quad (6.6.5)$$

### 6.6.2 Generalized DT-invariants for a Dynkin quiver

We will give a combinatorial proof of Theorem 6.6.1, which provides a slightly stronger statement.

Let  $p = \prod_{j=1}^l K_j^{\varepsilon_j} : \mathcal{H} \rightarrow \mathcal{H}'$  be a path (not necessarily directed) in  $\mathrm{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1])$ , where  $K_i$  are edges in  $\mathrm{EG}(Q)$  and the sign  $\varepsilon_j = \pm 1$  indicates the direction of  $K_j$  in  $p$ . Define the DT-function of  $p$  to be

$$\mathrm{DT}(Q; p) = \prod_{j=1}^l \mathbb{E}(y^{K_j})^{\varepsilon_j}.$$

Since we identify HN-strata with directed paths in Theorem 6.5.10, thus Theorem 6.6.1 can be rephrased as: the quantum dilogarithm of a directed path connecting  $\mathcal{H}_Q$  and  $\mathcal{H}_Q[1]$  is independent of the choice of the path. It is natural to ask if the path-independence holds for more general paths (not necessarily directed). The answer is yes within the subgraph  $\mathrm{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1])$ .

**Theorem 6.6.3.** *If  $p$  is a path in  $\text{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1])$ , then  $\text{DT}(Q; p)$  only depends on the head  $\mathcal{H}$  and tail  $\mathcal{H}'$  of  $p$ .*

*Proof.* We give a combinatorial proof. By Proposition 6.1.4,  $\pi_1(\text{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1]))$  is generated by the squares and pentagons as in (6.1.4). Thus we only need to check these two cases for the path-independence.

Notice that in the square or pentagon, we have  $\text{Hom}(S_i, S_j) = \text{Hom}(S_j, S_i) = 0$  and  $S_i, S_j \in \mathcal{H}_Q$ . In the square case we have

$$\text{Hom}^1(S_i, S_j) = \text{Hom}^1(S_j, S_i) = 0$$

and hence  $\langle \dim S_i, \dim S_j \rangle = \langle \dim S_j, \dim S_i \rangle = 0$  by (2.1.1), which implies

$$\mathbb{E}(y^{S_i}) \cdot \mathbb{E}(y^{S_j}) = \mathbb{E}(y^{S_j}) \cdot \mathbb{E}(y^{S_i}) \quad (6.6.6)$$

as required. In the pentagon case we have a triangle  $S_i \rightarrow T_j \rightarrow S_j \rightarrow S_i[1]$  and  $\dim S_i + \dim S_j = \dim T_j$ . Then

$$\text{Hom}^1(S_i, S_j) = 0, \quad \dim \text{Hom}^1(S_j, S_i) = 1$$

and hence  $\langle \dim S_i, \dim S_j \rangle = 0$  and  $\langle \dim S_j, \dim S_i \rangle = -1$  by (2.1.1). By the relations of the quantum affine space we have

$$\begin{aligned} y^{S_i} \cdot y^{S_j} &= q \cdot y^{S_j} \cdot y^{S_i}, \\ y^{T_j} &= q^{-\frac{1}{2}} \cdot y^{S_i} \cdot y^{S_j}. \end{aligned}$$

By the Pentagon Identity (see for example [32, Theorem 1.2]) we have

$$\mathbb{E}(y^{S_i}) \cdot \mathbb{E}(y^{S_j}) = \mathbb{E}(y^{S_j}) \cdot \mathbb{E}(y^{T_j}) \cdot \mathbb{E}(y^{S_i}) \quad (6.6.7)$$

as required. □

Therefore for any two heart  $\mathcal{H}_1, \mathcal{H}_2$  in  $\text{EG}(Q; \mathcal{H}, \mathcal{H}[1])$ , we have a *generalized DT-invariant*

$$\text{DT}(Q; \mathcal{H}_1, \mathcal{H}_2) := \text{DT}(Q; p) \quad (6.6.8)$$

where  $p$  is any path connecting  $\mathcal{H}$  and  $\mathcal{H}'$ . In particular, we have

$$\text{DT}(Q) = \text{DT}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1]). \quad (6.6.9)$$

## 6.7 Wall crossing formula for APR tilting

Let  $i$  be a sink in  $Q$  and  $\text{Sim } \mathcal{H}_Q = \{S_j\}_{j=1}^n$ . Then the APR-tilt  $\mathcal{H}_{Q'} = (\mathcal{H}_Q)_{S_i}^\sharp$  is a standard hearts in  $D(Q)$ , where  $Q'$  is obtained from  $Q$  by reversing the arrow at  $i$ . By Theorem 3.1.4, we have  $\text{Sim } \mathcal{H}_{Q'} = \{T_j\}_{j=1}^n$ , where  $T_i = S_i[1]$ ,  $T_j = \psi_{S_i}^\sharp(S_j)$  for  $j \neq i$  (and  $\psi$  is as in (3.1.1)). Let  $\dim'$  and  $\langle -, - \rangle'$  be the dimension vector and the Euler form, respectively, associated to  $Q'$ . Consider the quantum affine space  $\mathbb{A}_{Q'}$

$$\mathbb{Q}(q^{1/2})\langle z^S \mid S \in \text{Sim } \mathcal{H}_{Q'} \rangle / (z^{T_i} z^{T_j} = q^{\lambda_{Q'}(i,j)} z^{T_j} z^{T_i})$$

where  $z^S = z^{\dim' S}$  and

$$\lambda_{Q'}(i, j) = \langle T_j, T_i \rangle' - \langle T_i, T_j \rangle'.$$

By Theorem 6.6.3, we can also define DT-invariants  $\text{DT}(Q'; \mathcal{H}_1, \mathcal{H}_2)$  in  $\mathbb{A}_{Q'}$  for any  $\mathcal{H}_1, \mathcal{H}_2 \in \text{EG}(Q; \mathcal{H}_{Q'}, \mathcal{H}_Q[1])$ .

Notice that the labels of edges in  $\text{EG}(Q; \mathcal{H}_{Q'}, \mathcal{H}_Q[1])$  are in

$$\text{Ind}(\mathcal{H}_Q \cap \mathcal{H}_{Q'}) = \text{Ind } \mathcal{H}_Q - \{S_i\} = \text{Ind } \mathcal{H}_{Q'} - \{S_i[1]\}.$$

It is straightforward to check that the following conditions are equivalent

1°. for any hearts  $\mathcal{H}_1, \mathcal{H}_2 \in \text{EG}(Q; \mathcal{H}_{Q'}, \mathcal{H}_Q[1])$ ,

$$\text{DT}(Q; \mathcal{H}_1, \mathcal{H}_2) = \text{DT}(Q'; \mathcal{H}_1, \mathcal{H}_2),$$

2°. we have  $z^{T_i} = (y_i^S)^{-1}$  and  $z^M = y^M$  for any  $M \in \text{Ind}(\mathcal{H}_Q \cap \mathcal{H}_{Q'})$ .

3°. we have  $z^{T_i} = (y_i^S)^{-1}$  and  $z^{T_j} = y^{T_j}$  for  $j \neq i$ .

4°. we have  $z^{T_i} = (y_i^S)^{-1}$  and  $z^{S_j} = y^{S_j}$  for  $j \neq i$ .

Further, if the conditions above hold, the *wall crossing formula*

$$\text{DT}(Q) \cdot \mathbb{E}(y^{S_i})^{-1} = \mathbb{E}(y^{-S_i})^{-1} \cdot \text{DT}(Q') \quad (6.7.1)$$

comes for free because both sides equal to  $\text{DT}(Q; \mathcal{H}_{Q'}, \mathcal{H}_Q[1])$ .

## CHAPTER 7

## LINEAR CASE

Within this chapter, suppose  $Q$  is of  $A_n$  type. First, we will classify the hearts for  $A_n$  case; then we will calculate the stability spaces for  $A_2$  case. By abuse of notations, we will write  $\mathcal{D}(\Gamma_N A_n)$  and  $\text{EG}^\circ(\Gamma_N A_n)$  for  $\mathcal{D}(\Gamma_N Q)$  and  $\text{EG}^\circ(\Gamma_N Q)$  respectively.

### 7.1 Graded gentle tree

In [2], Assem and Happel gave the complete description of all iterated tilted algebras of type  $A_n$ , namely:

**Definition 7.1.1.** [3] Let  $A$  be an quiver algebra with acyclic quiver  $G$ . The algebra  $A \cong \mathbf{k}G/\mathcal{I}$  is called *gentle* if the bound quiver  $(G, \mathcal{I})$  has the following properties:

- 1°. Each point of  $G$  is the source and the target of at most two arrows.
- 2°. For each arrow  $\alpha \in (G)_1$ , there is at most one arrow  $\beta$  and one arrow  $\gamma$  such that  $\alpha\beta \notin \mathcal{I}$  and  $\gamma\alpha \notin \mathcal{I}$ .
- 3°. For each arrow  $\alpha \in (G)_1$ , there is at most one arrow  $\xi$  and one arrow  $\zeta$  such that  $\alpha\xi \in \mathcal{I}$  and  $\zeta\alpha \in \mathcal{I}$ .
- 4°. The ideal  $\mathcal{I}$  is generated by the paths in 3°.

If  $G$  is a tree, the gentle algebra  $A \cong \mathbf{k}G/\mathcal{I}$  is called a *gentle tree algebra*

**Theorem 7.1.2** (Assem-Happel [2]). *Let  $A$  be a quiver algebra with bound quiver  $(G, \mathcal{I})$ . Then  $A$  is an (iterated) tilted algebras of type  $A_n$  if and only if  $(G, \mathcal{I})$  is a gentle trees algebra.*

Considering the special properties of  $G$ , we can color its vertices with two colors, such that any two neighbor arrows  $\alpha, \beta$  have the same color if and only if  $\alpha\beta \in \mathcal{I}$  or

$\beta\alpha \in \mathcal{I}$ . Alternatively, we can also color it into two colors, such that any two neighbor arrows  $\alpha, \beta$  have the different colors if and only if  $\alpha\beta \in \mathcal{I}$  or  $\beta\alpha \in \mathcal{I}$ . By the properties above, either coloring is unique up to swapping colors. Hence we have another way to characterize the gentle tree algebra as follows.

**Definition 7.1.3.** A *gentle tree* is a quiver  $G$  with a 2-coloring, such that each vertex has at most one arrow of each color incoming or outgoing.

For a colored quiver  $G$ , there are two natural ideals

- $\mathcal{I}_G^+$ , generated by all unicolor-paths of length two;
- $\mathcal{I}_G^-$ , generated by all alternating color paths of length two.

**Proposition 7.1.4.** Let  $A = \mathbf{k}G/\mathcal{I}$  be a bound quiver algebra. We have the following equivalent statement:

- $A$  is a gentle tree algebra.
- $G$  is some gentle tree with  $\mathcal{I} = \mathcal{I}_G^+$  or  $\mathcal{I} = \mathcal{I}_G^-$ .

*Proof.* By one of the two ways of coloring, the relations in the ideal and the coloring of the gentle tree can be determined uniquely by each other.  $\square$

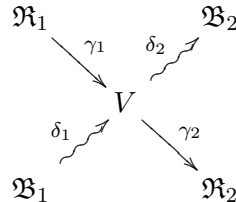
**Remark 7.1.5.** In fact, there is an irrelevant but interesting result that for a gentle tree  $G$ ,  $\mathbf{k}G/\mathcal{I}_G^+$  and  $\mathbf{k}G/\mathcal{I}_G^-$  are Koszul dual.

We are going to generalize Theorem 7.1.2 to describe all heart algebras in  $\mathcal{D}(A_n)$ .

**Definition 7.1.6.** A *graded gentle tree*  $\mathcal{G}$  is a gentle tree with a positive integer attached to each arrow as degree/grading. Moreover, its (graded) colored path algebra  $\mathcal{A}(\mathcal{G})$  to be  $\mathbf{k}\mathcal{G}/\mathcal{I}_{\mathcal{G}}^-$ .

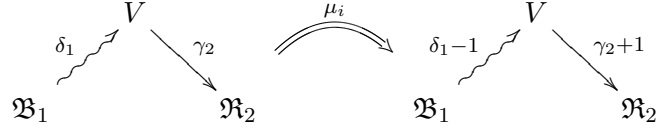
Define a mutation  $\mu$  on graded gentle tree as follow.

**Definition 7.1.7.** For a graded gentle tree  $\mathcal{G}$ , Let  $V$  be a vertex with neighborhood

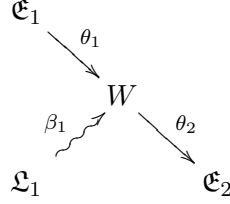


where  $\mathfrak{R}_i, \mathfrak{B}_i$  are the sub trees and  $\gamma_i, \delta_i$  are degrees,  $i = 1, 2$ . The straight lines represent one color and the curly lines represent the other color. Define the *mutation*  $\mu_V$  on vertex  $V$  as:

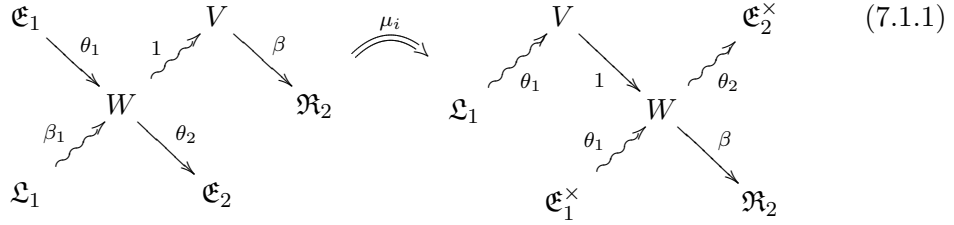
- If  $\delta_1 \geq 1$ , the mutation on the lower part of the quiver is:



- If  $\delta_1 = 1$ , represent  $\mathfrak{B}_1$  as



and the mutation on the lower part of the quiver is:



where  $\mathfrak{X}^\times$  is the operation of swapping colors on a graded gentle trees  $\mathfrak{X}$ .

- the mutation of the upper part follows the same rule as the lower part.

Dually, define the mutation  $\mu_V^{-1}$  to be the reverse of  $\mu_V$  (which follows the similar rules).

Clearly, the set of all graded gentle trees with  $n$  vertices is closed under such mutation. We will show that there is a one-one correspondence between all graded gentle tree with  $n$  vertices and all heart algebras in  $\mathcal{D}(A_n)$ .

Taking colored path algebra commutes with the mutations in the following sense.

**Proposition 7.1.8.** *Let  $\mathcal{G}$  be a graded gentle tree. If  $\mathcal{A}(\mathcal{G})$  is a heart algebra  $\mathcal{E}_{\mathcal{H}}$  for some  $\mathcal{H} \in \text{EG}(A_n)$  and vertex  $V$  corresponds to the simple  $S \in \text{Sim } \mathcal{H}$ , then  $\mathcal{E}_{\mathcal{H}_S^\#} = \mathcal{A}(\mu_V \mathcal{G})$ .*

*Proof.* Using Theorem 4.3.2 the proposition follows by a direct calculation.  $\square$

**Theorem 7.1.9.** *The heart algebras of the hearts in  $\mathcal{D}(A_n)$  are precisely the colored path algebras of graded gentle trees with  $n$  vertices.*

*Proof.* Without loss of generality, let  $Q$  has straight orientation. Then any heart in  $\mathcal{D}(A_n)$  can be iterated tilted from the standard heart  $\mathcal{H}_Q$ , by Theorem 4.2.5. Moreover,

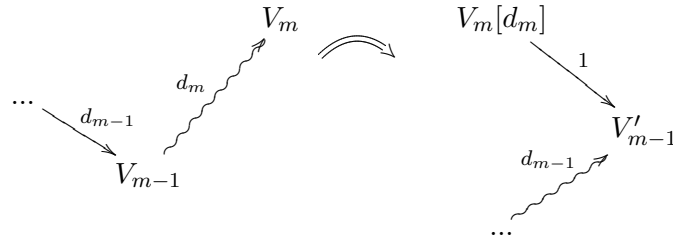
the heart algebra of  $\mathcal{H}_Q$  is the colored path algebra of the graded gentle tree with  $n$  vertices and without unicolor-path of length two. Starting from  $\mathcal{H}_Q$ , the theorem follows inductively from the commutativity in Proposition 7.1.8.  $\square$

## 7.2 An application

Using graded gentle tree, we can give a classification of all indecomposable (up to shift) in  $\mathcal{D}(A_n)$  with respect to a fixed heart. Let  $\mathcal{H}$  be a heart in  $\mathcal{D}(A_n)$  with heart algebra  $\mathcal{E}_{\mathcal{H}} = \mathcal{A}(\mathcal{G})$  for some graded gentle tree  $\mathcal{G}$ . Let  $\text{Sim } \mathcal{H} = \{S_1, \dots, S_n\}$  and correspond to the vertices  $\{V_1, \dots, V_n\}$  of  $\mathcal{G}$ .

**Lemma 7.2.1.** *For any ordered pair of vertices  $V, V'$  (not necessary different) in  $\mathcal{G}$ , there is an unique sequence of unicolor-paths  $p_1, p_2, \dots, p_m$  connecting them, such that  $\{h(p_i), t(p_i)\} = \{V_{i-1}, V_i\}$ ,  $V_0 = V, V_m = V'$  and their colors are alternative. Let  $s_0 = 0$ ,  $s_j = \sum_{i=1}^j (\deg p_i - 1)$ ,  $j = 1, \dots, m$ . Then there is an indecomposable  $M$  in  $\mathcal{D}(A_n)$  which admits a filtration of triangles with factors  $\{S_i[s_i] \mid i = 0, \dots, m\}$ .*

*Proof.* By definition of gentle tree, such unicolor-paths uniquely exist. To show there is such indecomposable  $M$ , use induction on  $m$ . If  $m = 0$ , then  $M = S_0$ . Now suppose that  $m > 0$ . Let  $S_i$  be the simples corresponding to  $V_i$ . Without loss of generality, suppose  $h(p_m) = V_m$ . Consider the heart  $\mathcal{H}_S^{d_m\#}$  and  $\mathcal{G}' = \mu_{V'}^{d_m}(\mathcal{G})$ . Notice that the alternating unicolor-path  $p_1, \dots, p_{m-1}$  remains in  $\mathcal{G}'$  but with different colors. Moreover  $V_{m-1}$  corresponds to a new simple in  $\mathcal{H}_S^{d_m\#}$ , which is the extension of  $S_{m-1}$  on top of  $S[d_m - 1]$ .



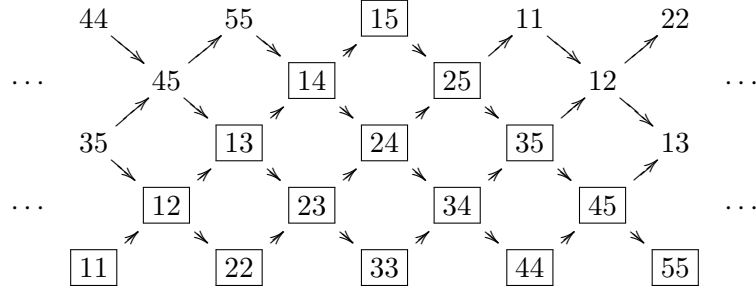
By induction, we know that there is an indecomposable  $M$  corresponds to  $V$  and  $V_{m-1}$  (in  $\mathcal{G}'$ ), which corresponds to  $V$  and  $V_m$  (in  $\mathcal{G}$ ) by a direct checking of the shifts of  $S_m$ .  $\square$

By the construction in Lemma 7.2.1 we have a map  $\psi : \mathcal{G}_0 \times \mathcal{G}_0 \rightarrow \text{Ind } \mathcal{D}(A_n)$ .

**Corollary 7.2.2.** *The map  $\psi$  induces a bijection*

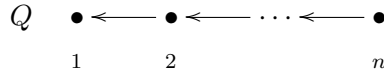
$$\tilde{\psi} : \mathcal{G}_0^2 \rightarrow \text{Ind } \mathcal{D}(A_n)/[1]$$

where  $\mathcal{G}_0^2$  is the set of all unordered pair of vertices in  $\mathcal{G}_0$ .

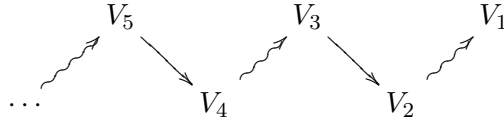

 Figure 7-1:  $\mathcal{D}(A_5)$  with a standard heart

*Proof.* Notice that the domain and range both have size  $\binom{n+1}{2}$ . Then the bijectivity is equivalent to the injectivity which follows from checking the Grothendieck group  $\mathcal{K}(\mathcal{D}(A_n))$ .  $\square$

**Example 7.2.3.** For the standard heart which corresponds to the straight orientation of  $A_n$ :

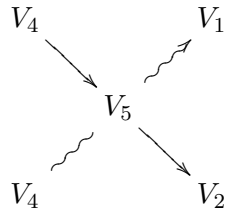


we have  $\text{Ind } \mathcal{H}_Q = \{M_{ij} \mid 1 \leq i \leq j \leq n\}$ , where  $[M_{ij}] = [S_i] + \dots [S_j]$  in the Grothendieck group  $\mathcal{K}(\mathcal{D}(A_n))$ . The corresponding (graded) gentle tree is



Then  $M_{ij} = \psi(V_i, V_j)$ . In Figure 7-1,  $ij$  represent an indecomposable in  $\tilde{\psi}(V_i, V_j)$ , the boxed ones are the ones in the heart and the simples are the boxed ones with  $i = j$ .

**Example 7.2.4.** For the following (graded) gentle tree

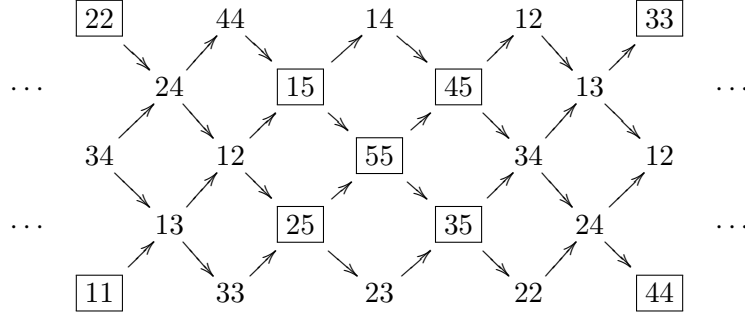


we have the corresponding  $\mathcal{D}(A_5)$  as in Figure 7-2 (notation is the same as above) .

### 7.3 Graded gentle cycles

Let  $\mathcal{G}$  be a graded gentle tree. Define the *degree* of a graded gentle tree be the maximal degree of maximal unicolor-path in  $\mathcal{G}$ . Equivalently, it is the maximal integer that can





**Figure 7-2:**  $\mathcal{D}(A_5)$  with a nonstandard heart

not strictly bound its colored path algebra.

**Definition 7.3.1.** A *graded gentle cycle*  $\mathcal{G}^\circ$  of degree  $N$  is the *cyclic- $N$  completion* of some graded gentle tree  $\mathcal{G}$  with  $\deg \mathcal{G} < N$ , that is, the graded quiver obtained from  $\mathcal{G}$  by completing every maximal unicolor-path  $p : i \rightarrow j$  in  $\mathcal{G}$  to a cycle  $c$  with an arrow  $a : j \rightarrow i$  with degree  $\deg a = N - \deg p$ . Denote this completion by  $\Theta_N$ , i.e.  $\mathcal{G}^\circ = \Theta_N(\mathcal{G})$ . Moreover, a *cut* of a graded gentle cycle  $\mathcal{G}^\circ$  is to cut it into a graded gentle tree, i.e. deleting (any) one arrow in each unicolor cycle. Notice that, if  $\deg \mathcal{G}^\circ = N$ , then it is the cyclic- $N$  completion of any its cut.

A direct calculation gives the following lemma.

**Lemma 7.3.2.** Let  $\mathcal{G}^\circ$  be a graded gentle cycles and  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two of its cuts. Then we have  $T_N(\mathcal{A}(\mathcal{G}_1)) = T_N(\mathcal{A}(\mathcal{G}_2))$ .

Thus, we can define the (graded) colored path algebra  $\mathcal{A}(C)$  of a graded gentle cycle  $C$  to be  $T_N(\mathcal{A}(\mathcal{G}))$  for any its cut  $\mathcal{G}$ .

**Theorem 7.3.3.** The heart algebras of hearts in  $\text{EG}^\circ(\Gamma_N A_n)$  are precisely the colored path algebras of graded gentle cycles.

*Proof.* Let  $\mathcal{H} \in \text{EG}^\circ(\Gamma_N A_n)$ . By Theorem 5.2.1,  $\mathcal{H}$  is induced from some heart  $\widehat{\mathcal{H}} \in \text{EG}(A_n)$  via some strong immersion, which implies

$$\mathcal{E}_{\mathcal{H}} = T_N(\mathcal{E}_{\widehat{\mathcal{H}}}).$$

By Theorem 7.1.9,  $\mathcal{E}_{\widehat{\mathcal{H}}} = \mathcal{A}(\mathcal{G})$  for some graded gentle tree  $\mathcal{G}$ . Thus we have  $\mathcal{A}(\mathcal{G}^\circ) = T_N(\mathcal{A}(\mathcal{G})) = \mathcal{E}_{\mathcal{H}}$ , where  $\mathcal{G}^\circ = T_N(\mathcal{G})$ .  $\square$

## 7.4 Construction of $A_2$ -type exchange graph via the Farey graph

Let  $Q$  be an  $A_2$ -type quiver within this subsection.

There are many descriptions of the Farey graph FG including the following:

- It is the curve complex  $\mathcal{M}_{0,4}$  of a 4-punctured sphere. More precisely, its vertex set corresponds to homotopy classes of (simple closed) curves on the sphere  $\mathcal{M}_{0,4}$  and there are an edge between two vertices if and only if the corresponding curves have intersection number two.
- It is the curve complex  $\mathcal{M}_{1,1}$  of a (1-)punctured torus. More precisely, its vertex set corresponds to homotopy classes of (simple closed) curves on the sphere  $\mathcal{M}_{1,1}$  and there are an edge between two vertices if and only if the corresponding curves have intersection number one.
- Its vertex set is  $\text{FG}_0 = \mathbb{Q} \cup \{\infty\}$ . And there is an edge  $(p/q - r/s)$  (we assume  $\infty = 1/0$  in this notation) if and only if  $|ps - rq| = 1$ .

Notice that the first two interpretations of the Farey graph is related to the mirror category of  $\mathcal{D}(\Gamma_N Q)$  (see [40]). Hence the description of exchange graphs below provides some more evidence for Kontsevich's homological mirror symmetry (in this basic case).

#### 7.4.1 Spherical objects via vertices in FG

Denote by  $\text{Sph}(\Gamma_N A_2)$  the set of all spherical objects which are simples in some hearts in  $\text{EG}^\circ(\Gamma_N A_2)$ . An element in  $\text{Sph}(\Gamma_N A_2)/[1]$  is a shift orbit of some spherical object  $X \in \text{Sph}(\Gamma_N A_2)$  and will be denoted by  $X[ ]$ . Then there is a unique bijection

$$\chi : \text{FG}_0 \xrightarrow{\sim} \text{Sph}(\Gamma_N A_2)/[1]$$

determined by the following conditions:

- 1°.  $\chi(0) = X_0[ ]$  and  $\chi(\infty) = X_\infty[ ]$  where  $X_0$  and  $X_\infty$  are the simples in the standard heart  $\mathcal{H}_\Gamma$  satisfying  $\text{Ext}^1(X_0, X_\infty) \neq 0$ .
- 2°. For a clockwise triangle  $\triangle = (a, b, c)$  in FG, there exists a triangle

$$X_a \rightarrow X_b \rightarrow X_c \rightarrow X_a[1] \tag{7.4.1}$$

in  $\mathcal{D}(\Gamma_N A_2)$  such that  $X_j$  is in the shift orbit  $\chi(j)$  for  $j = a, b, c$  and

$$X_a = \phi_{X_b}(X_c)[-1], \quad X_b = \phi_{X_c}(X_a), \quad X_c = \phi_{X_a}(X_b).$$

For instance, we have

$$\chi(1) = \phi_{X_0}(X_\infty)[ ], \quad \chi(-1) = \phi_{X_\infty}(X_0)[ ]$$

where  $\pm 1$  are the other vertices in the triangle containing the edge  $(\infty - 0)$ .

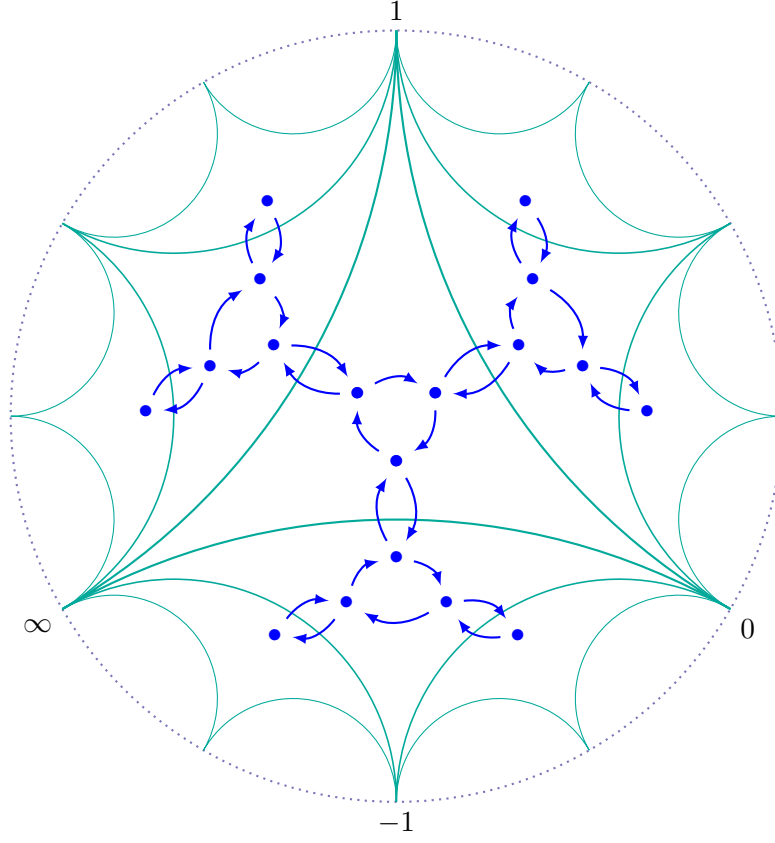


Figure 7-3: the Farey graph FG with associated (directed) graph  $\mathcal{G}_3$

### 7.4.2 L-immersions via triangles in FG

The second condition that we impose on  $\chi$  provides us the following property:

- every triangle  $\Delta$  in FG induces a unique L-immersions  $F_\Delta$  up to the image.

More precisely, let  $\text{Ind mod } \mathbf{k}A_2 = \{Y_a, Y_b, Y_c\}$  with short exact sequence  $0 \rightarrow Y_a \rightarrow Y_b \rightarrow Y_c \rightarrow 0$ . Then we have a L-immersion  $F_\Delta$  defined by

$$F_\Delta(Y_j) = X_j, \quad j = a, b, c$$

where  $X_j$  are as in (7.4.1). We also have

$$F_\Delta(\text{Ind } \mathcal{D}(A_2)) = X_a[\ ] \cup X_b[\ ] \cup X_c[\ ],$$

where we consider  $X_j[\ ]$  as  $\{X_j[m]\}_{m \in \mathbb{Z}}$  here. This justifies the uniqueness of the L-immersion as we claimed.

### 7.4.3 Directed graph associated to FG

Let  $N = 3$  in this subsection. Define an infinite (directed) graph  $\mathcal{G}_3$  with a grading function

$$\text{gr} : E(\mathcal{G}_3) \rightarrow \mathbb{Q}$$

on the edge set  $E(\mathcal{G}_3)$  as follows:

- To each triangle  $\Delta$  in FG we associate a clockwise oriented triangle  $T_\Delta$  whose vertices correspond to the edges of  $\Delta$ .
- Hence to each edge  $\Lambda$  of FG there will be associated two vertices; connect them with a pair of edges forming a 2-cycle  $C_\Lambda$

$$C_\Lambda : v_1 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} v_2 \quad (7.4.2)$$

- $\mathcal{G}_3$  is the union of all  $T_\Delta$  and  $C_\Lambda$ .
- Define the grading by

$$\text{gr}(e) = \begin{cases} \frac{1}{3}, & \text{if } e \in T_\Delta, \\ \frac{1}{2}, & \text{if } e \in C_\Lambda. \end{cases}$$

We have a bijection

$$\nu : \mathcal{G}_3 \xrightarrow{\sim} \text{EG}(\Gamma_3 A_2)/[1].$$

To see this, choose a clockwise triangle  $\Delta = (a, b, c)$  in FG with corresponding triangle  $(T_a, T_b, T_c)$  in  $\mathcal{G}_3$  and  $\{X_a, X_b, X_c\}$  as in (7.4.1) for instance. Then we have

$$\nu(T_a, T_b, T_c) = (\mathcal{H}_a[ ], \mathcal{H}_b[ ], \mathcal{H}_c[ ]) \quad (7.4.3)$$

where the hearts are determined by their simples as follow

$$\text{Sim } \mathcal{H}_a = \{X_c[-1], X_b\}, \quad \text{Sim } \mathcal{H}_b = \{X_a, X_c\}, \quad \text{Sim } \mathcal{H}_c = \{X_b, X_a[1]\}.$$

Furthermore, there exists a unique  $\mathbb{Z}$ -cover  $\pi : \widetilde{\mathcal{G}}_3 \rightarrow \mathcal{G}_3$  (with respect to the grading) satisfying

- $\widetilde{\mathcal{G}}_3$  sits inside  $\mathcal{G}_3 \times \frac{1}{6}\mathbb{Z}$  and  $(\nu^{-1}(\mathcal{H}_\Gamma[ ]), 0)$  is in  $\widetilde{\mathcal{G}}_3$ .
- For any  $v \in \mathcal{G}_3$  we have  $\pi^{-1}(v) = \{v\} \times \mathbf{N}_v$ , where  $\mathbf{N}_v$  is a coset of  $\mathbb{Z}$  in  $\frac{1}{6}\mathbb{Z}$ .
- Any lifting  $\tilde{e}$  in  $\widetilde{\mathcal{G}}_3$  of an edge  $e$  in  $\mathcal{G}$  has an additional vertical shift by  $\text{gr}(e)$ .

Notice that  $\widetilde{\mathcal{G}}_3$  can be covered by countably many oriented pentagons which sit between vertices  $(v, t)$  and  $(v, t + 1)$  for each  $(v, h)$  in  $\widetilde{\mathcal{G}}_3$ . More precisely, such a pentagon (see Figure 7-4) consist of exactly a lift of some triangle  $T_\Delta$  and a lift of some 2-cycle  $C_\Lambda$ .

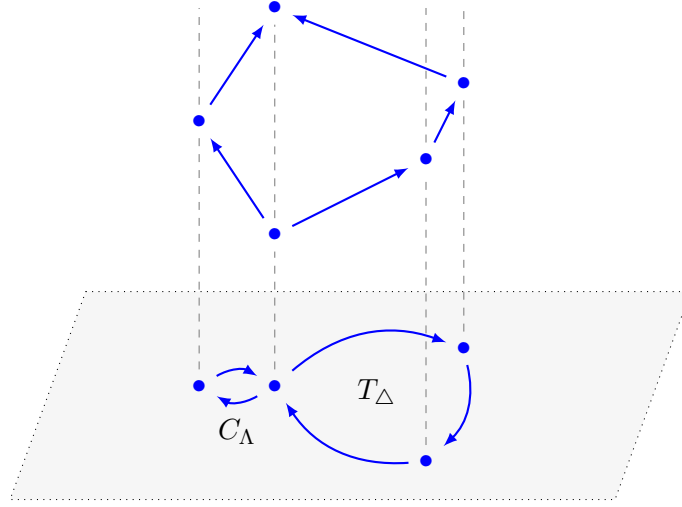


Figure 7-4: A lifting pentagon

Now we have a bijection

$$\tilde{\nu} : \widetilde{\mathcal{G}}_3 \xrightarrow{\sim} \text{EG}(\Gamma_3 A_2).$$

To see this, consider  $T_\Delta = (\infty, 1, 0)$  with corresponding (7.4.3) for instance. Then we have

$$\tilde{\nu}(T_\infty, m - \frac{1}{3}) = \mathcal{H}_\infty[m], \quad \tilde{\nu}(T_1, m) = \mathcal{H}_1[m], \quad \tilde{\nu}(T_0, m + \frac{1}{3}) = \mathcal{H}_0[m].$$

#### 7.4.4 Variations in general case

In general define  $\mathcal{G}_N$  to be a deformation of  $\mathcal{G}_3$  for  $N > 1$  (c.f. Figure 7-7).

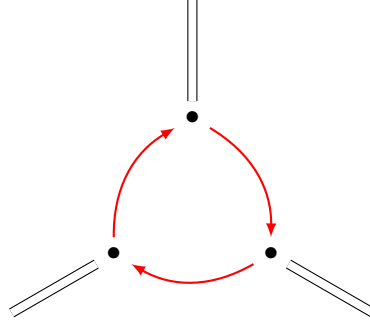
More precisely, replace each 2-cycle  $C_\Lambda$  (7.4.2) in  $\mathcal{G}_3$  by a chain  $C_\Lambda(N)$  of  $(N - 2)$  2-cycles

$$C_\Lambda(N) : v_1 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} v_2 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \dots \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} v_{N-2} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} v_{N-1}.$$

In the special case when  $N = 2$ ,  $C_\Lambda(2)$  collapses to a vertex. Then the  $\mathbb{Z}$ -cover  $\widetilde{\mathcal{G}}_N$  of  $\mathcal{G}_N$  which is uniquely determined by the same condition as above is isomorphic to  $\text{EG}^\circ(\Gamma_N A_2)$ .

In summary, we construct the following bijections:

$$\begin{cases} \chi : \text{FG}_0 \xrightarrow{\sim} \text{Sph}(\Gamma_N A_2)/[1], \\ \nu : \mathcal{G}_N \xrightarrow{\sim} \text{EG}^\circ(\Gamma_N A_2)/[1], \\ \tilde{\nu} : \widetilde{\mathcal{G}}_N \xrightarrow{\sim} \text{EG}^\circ(\Gamma_N A_2). \end{cases} \quad (7.4.4)$$


 Figure 7-5:  $\text{EG}^o(A_2)/[1]$ 

### 7.4.5 Related examples

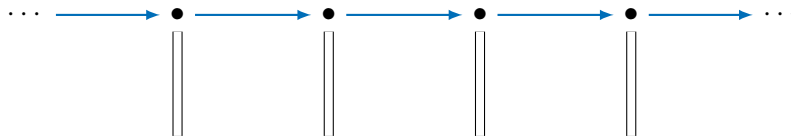
An  $\infty$ -chain of 2-cycle  $C_\infty(v)$  with base point  $v$  consists of vertices  $\{v = v_0, v_1, v_2, \dots\}$  such that there are two edges between  $v_j$  and  $v_{j+1}$  forming a 2-cycle as follows:

$$C_\infty(v) : v_0 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} v_1 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} v_2 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \dots$$

Define the grading of any edge in any  $\infty$ -chain of 2-cycle to be  $\frac{1}{2}$ .

The exchange graph  $\text{EG}^o(A_2)/[1]$  consists of 3  $\infty$ -chain of 2-cycles  $C_\infty(u_1)$ ,  $C_\infty(u_2)$  and  $C_\infty(u_3)$  while their base points  $u_i$  form a 3-cycle  $\Delta_u$  (see Figure 7-5, where  $(\bullet \text{ --- } \bullet)$  denotes  $C_\infty(v)$ ). Let the edges in  $\Delta_u$  have grading  $\frac{1}{3}$ . The exchange graph  $\text{EG}^o(A_2)$  is the unique  $\mathbb{Z}$ -covering of  $\text{EG}^o(A_2)/[1]$  with respect to the grading.

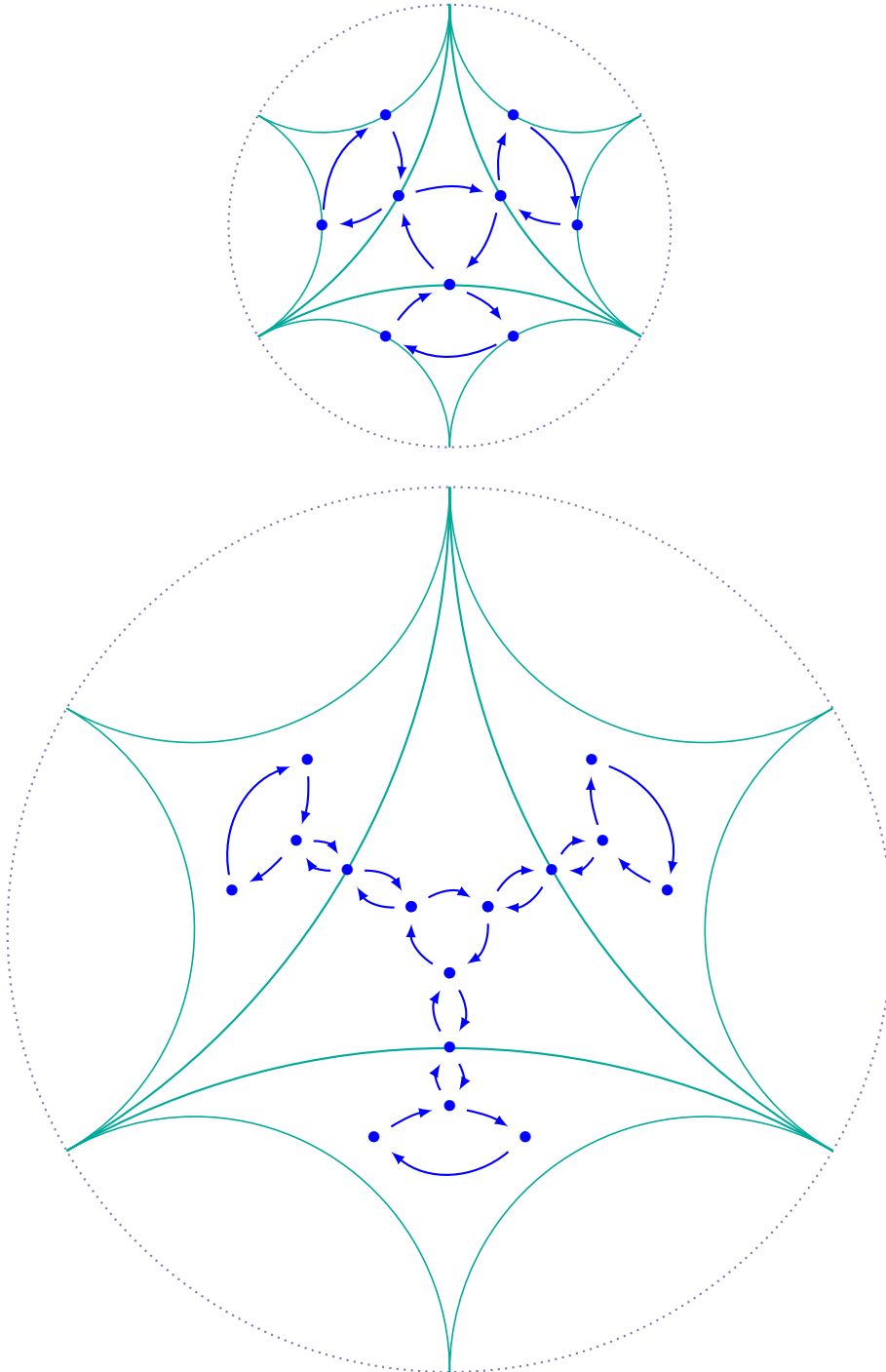
Consider the Kronecker quiver  $K : (2 \rightrightarrows 1)$ . Let the principal component of the exchange graph of  $\mathcal{D}(K)$  be  $\text{EG}^o(K)$ . Then  $\text{EG}^o(K)/[1]$  consists of a family of  $\infty$ -chain of 2-cycles  $\{C_\infty(w_j)\}_{j \in \mathbb{Z}}$  while their base points  $w_j$  form a directed line  $L_w$  (see Figure 7-6). Let the edges in  $L_w$  have grading 0. The exchange graph  $\text{EG}^o(K)$  is the unique  $\mathbb{Z}$ -covering of  $\text{EG}^o(K)/[1]$  with respect to the grading.


 Figure 7-6:  $\text{EG}^o(K)/[1]$ 

## 7.5 Stability Spaces for two vertex quivers

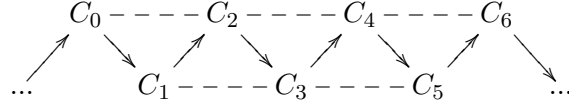
### 7.5.1 Stability Space for A2-type quiver

Let  $Q$  be the quiver of type  $A_2$  with orientation  $2 \rightarrow 1$  and  $\text{Ind } \mathcal{H}_Q = \{C_1, C_2, C_3\}$  such that  $\text{Ext}^1(C_3, C_1) \neq 0$ . Denote  $C_i[m]$  by  $C_{3m+i}$ . The AR-quiver of  $\mathcal{D}(A_2)$  is isomorphic



**Figure 7-7:** *Two variations of  $\mathcal{G}_3$ :  $\mathcal{G}_2$  and  $\mathcal{G}_4$*

to the translation quiver  $\mathbb{Z}A_2$  as follows



The dashed lines denote the  $\tau$ -orbits where  $\tau$  is the AR-functor (see, e.g. [3]). We have

$$\text{Aut } \mathcal{D}(A_2) \simeq \mathbb{Z}\langle \xi \rangle,$$

where the generator  $\xi = \tau \circ [1]$  satisfying  $\xi(C_j) = C_{j+1}$  and  $\xi^3 = [1]$ .

**Lemma 7.5.1.** *Let  $\sigma = (Z, \mathcal{P})$  be a stability condition in  $\mathcal{D}(A_2)$ . There exists an element  $\zeta \in \text{Aut } \mathcal{D}(A_2)$  and an nonnegative integer  $m$  such that the simples in the heart of  $\zeta \circ \sigma$  are  $C_1$  and  $C_3[m]$ . In particular, there are three types of stability conditions on  $\mathcal{D}(A_2)$ :*

- *Every indecomposable object is stable.*
- *Up to shift, two indecomposables are stable and one is semistable (but not stable).*
- *Up to shift, two indecomposables are stable and one is not semistable.*

*Proof.* Notice that  $\text{EG}(Q)$  is connected. By Theorem 3.1.4, we know the changes of simple during tilting, and then The first assertion follows by direct calculating. By comparing the phases of  $C_1$  and  $C_3$  with respect to the stability condition  $\xi \circ \sigma$ , we get the three cases.  $\square$

Let

$$\begin{aligned} \tilde{U} &= \{(Z, \mathcal{P}) \in \text{Stab}(A_2) \mid C_j \text{ are stable for } j = 1, 2, 3\}, \\ \tilde{W}_j &= \{(Z, \mathcal{P}) \in \text{Stab}(A_2) \mid C_j \text{ is not semistable}\}, \quad j = 1, 2, 3. \end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned} \partial \tilde{W}_j &= \{(Z, \mathcal{P}) \in \text{Stab}(A_2) \mid C_j \text{ is semistable but not stable}\}, \\ \partial \tilde{U} &= \partial \tilde{W}_1 \cup \partial \tilde{W}_2 \cup \partial \tilde{W}_3, \\ \text{Stab}(A_2) &= \tilde{U} \cup \partial \tilde{U} \cup \tilde{W}_1 \cup \tilde{W}_2 \cup \tilde{W}_3. \end{aligned}$$

Notice that the intersection of  $\mathbb{C}$ -actions and  $\text{Aut } \mathcal{D}(A_2)$  is  $\mathbb{Z}$  with generator  $-1 \in \mathbb{C}$  or



$[1] \in \text{Aut } \mathcal{D}(A_2)$ . Therefore we have a commutative diagram:

$$\begin{array}{ccc}
 & \text{Stab}(A_2) & \\
 \swarrow / \text{Aut} & & \searrow / \mathbb{C} \\
 \mathcal{M}_A & & \mathcal{M}_C \\
 \searrow / \mathbb{C}^* & & \swarrow / c_3 \\
 & \mathcal{M} &
 \end{array} \tag{7.5.1}$$

where  $\mathcal{C}_3 = \text{Aut} / \mathbb{Z}[1]$ ,  $\mathbb{C}^* = \mathbb{C} / \mathbb{Z}$  and  $\mathcal{M} = \text{Aut } \mathcal{D}(A_2) \setminus \text{Stab}(A_2) / \mathbb{C}$ . Let  $U, W_j \subset \mathcal{M}_C$  be the quotient spaces of  $\tilde{U}$  and  $\tilde{W}_j$  in  $\mathcal{M}_C$  respectively. We have a conformal isomorphism  $f : R \rightarrow \overline{W_2} \cup U$  (see Figure 7-8), where

$$R = \{\Theta = x + y\mathbf{i} \mid x < 1\} \subset \mathbb{C} \tag{7.5.2}$$

such that  $f(\Theta) = [\sigma]$  in  $\mathcal{M}_C$  and the stability condition  $\sigma = (Z, \mathcal{P})$  is determined by the following conditions

- $Z(C_1) = 1$  and  $Z(C_3) = \exp(i\pi\Theta)$ ;
- The simples in the heart of  $\sigma$  are  $C_1$  and  $C_3[m]$ , where  $m = -\lfloor \text{Im } \Theta \rfloor$ .

Let  $V = f^{-1}(U)$  and  $V_2 = f^{-1}(W_2)$ . Denote  $\mathbb{T}$  the triangle with vertices  $T_1 = 1, T_2 = 0$  and  $T_3 = -Z(C_3)$ . The  $\mathcal{C}_3$ -action on  $U$  will identify the stability conditions whose corresponding triangles  $\mathbb{T}$  are similar to each other. The red lines  $l_i$  in Figure 7-8 correspond to the case when  $\mathbb{T}$  is an isosceles triangle (with vertex angle at  $T_i$ ), where

$$\begin{cases}
 l_1 = \{\Theta = x + y\mathbf{i} \mid x \in (\frac{1}{2}, \frac{2}{3}], y\pi = -\ln(-2\cos x\pi)\}; \\
 l_2 = \{\Theta = x + y\mathbf{i} \mid y = 0, x \in [\frac{2}{3}, 1)\}. \\
 l_3 = \{\Theta = x + y\mathbf{i} \mid x \in (\frac{1}{2}, \frac{2}{3}], y\pi = \ln(-2\cos x\pi)\}.
 \end{cases}$$

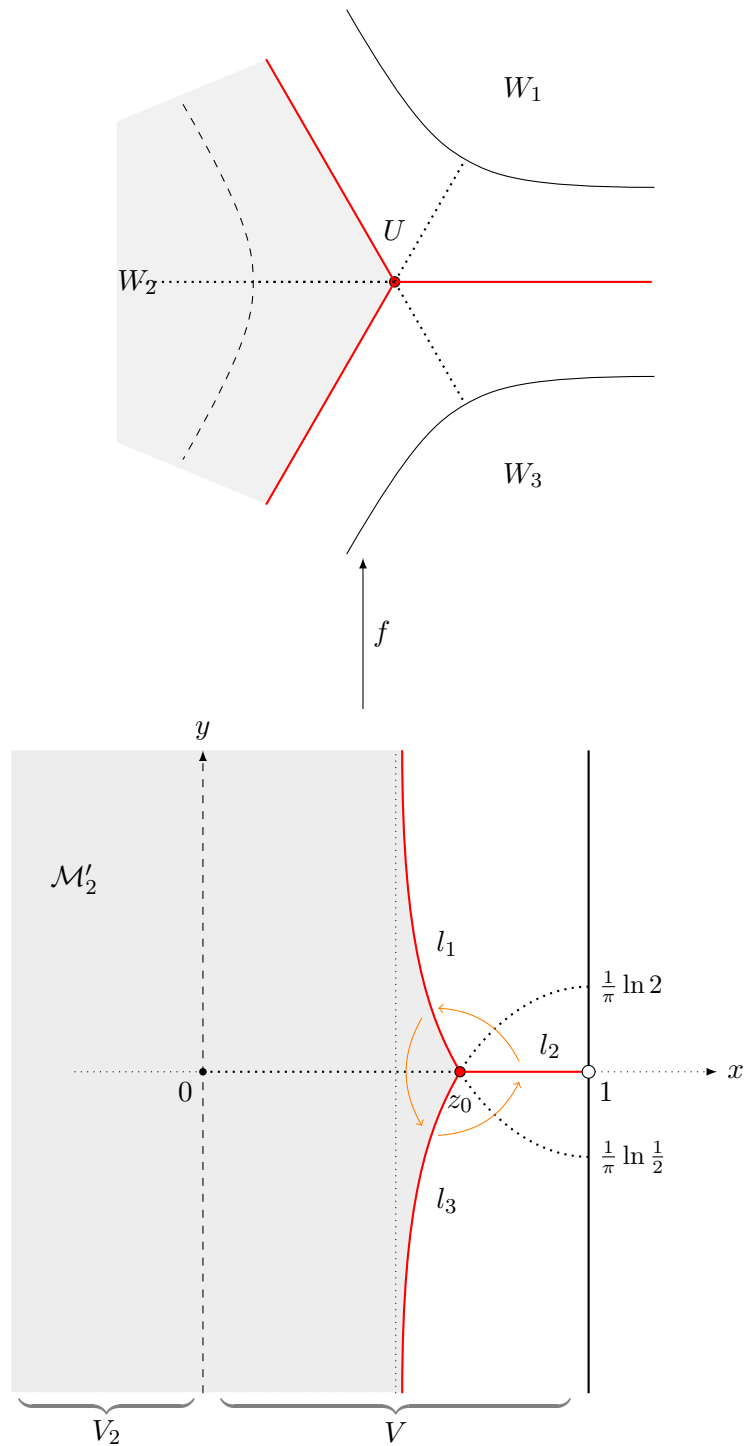
Moreover let  $\omega_0 : \mathcal{M}_C \rightarrow \mathcal{M}_C$  be the conformal map with order 3 corresponding to the  $\mathcal{C}_3$ -action and sending  $W_j$  to  $W_{j+1}$ . Also denote by  $\omega_0$ , the induced  $\mathcal{C}_3$ -action on  $V$ . Denote  $\mathcal{M}'_2$  the region strictly right bounded by  $l_1 \cup l_3$  in Figure 7-8. Then  $\mathcal{M}_2 = f(\mathcal{M}'_2)$  is a fundamental domain for the quotient map  $\mathcal{M}_C \rightarrow \mathcal{M}$ .

**Lemma 7.5.2.**  *$\mathcal{M}$  can be obtained from  $\overline{\mathcal{M}'_2}$  by identifying the points on the boundary  $l_1 \cup l_3$  with respect to the reflection of  $x$ -axis, Moreover,  $z_0 = l_1 \cap l_2 \cap l_3 = \frac{2}{3}$  is the only orbitfold point in  $\partial\mathcal{M}$ , which is with order  $\frac{1}{3}$ .*

*Proof.* The lemma follows from the facts that  $\omega_0(l_j) = l_{j+1}$  and  $l_1 \cap l_2 \cap l_3 = \{0\}$ .  $\square$

Let  $\mathcal{M}_3 = \omega_0(\mathcal{M}_2)$  and  $\mathcal{M}_1 = \omega_0(\mathcal{M}_3)$ . By Lemma 7.5.2, we have

$$\mathcal{M}_C = \overline{\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3} \quad \text{and} \quad \overline{\mathcal{M}_{j-1}} \cap \overline{\mathcal{M}_j} = f(l_{j+1}).$$



**Figure 7-8:** The conformal isomorphism  $f : R \rightarrow \overline{W_2} \cup U$

**Lemma 7.5.3.** *We have a conformal isomorphism  $g : \mathcal{M}_C \xrightarrow{\cong} \mathbb{C}$ .*

*Proof.* Let  $l(j) = \{z \in \mathbb{C} \mid \arg z = \frac{2\pi}{3}j\}$ . Using Riemann mapping theorem and Reflection Principle (as in [45, Lemma 4.4]), we have a map  $g'_2$  sending  $\overline{\mathcal{M}'_2}$  conformally isomorphic to

$$\mathcal{M}(2) = \{z \in \mathbb{C} \mid \arg z \in [\frac{2\pi}{3}, \frac{4\pi}{3}]\},$$

such that  $g'_2(\bar{z}) = \overline{g'_2(z)}$ . Let  $g_2 = g'_2 \circ f^{-1}$ , then we have  $g_2 : \mathcal{M}_2 \xrightarrow{\cong} \mathcal{M}(2)$ . Define  $\omega : \mathbb{C} \rightarrow \mathbb{C}$  by  $\omega(z) = z \cdot \exp(\frac{2\pi}{3}\mathbf{i})$  and let

$$\mathcal{M}(3) = \omega(\mathcal{M}(2)), \quad \mathcal{M}(1) = \omega(\mathcal{M}(3)).$$

Then we have two conformal isomorphisms

$$\begin{aligned} g_1(z) &= \omega^{-1} \circ g_2 \circ \omega_0 : \mathcal{M}_1 \xrightarrow{\cong} \mathcal{M}(1), \\ g_3(z) &= \omega \circ g_2 \circ \omega_0^{-1} : \mathcal{M}_3 \xrightarrow{\cong} \mathcal{M}(3). \end{aligned}$$

By [12, Theorem 11-8], we can conformally extend  $g_j$  to the smooth boundary

$$f(l_{j-1} \cup l_{j+1} - \{z_0\})$$

such that  $g \circ f(l_{j\pm 1}) = l(j \pm 1)$ . Notice that the extended maps  $g_1, g_2$  and  $g_3$  agree on

$$f(l_1 \cup l_2 \cup l_3 - \{z_0\})$$

by a direct calculation, thus we obtain a conformally isomorphism

$$g : \mathcal{M}_C - \{f(z_0)\} \longrightarrow \mathbb{C} - \{0\}.$$

Then by [12, Theorem 11-8] again, we can conformally extend  $g$  to the boundary  $\{f(z_0)\}$  which implies the lemma.  $\square$

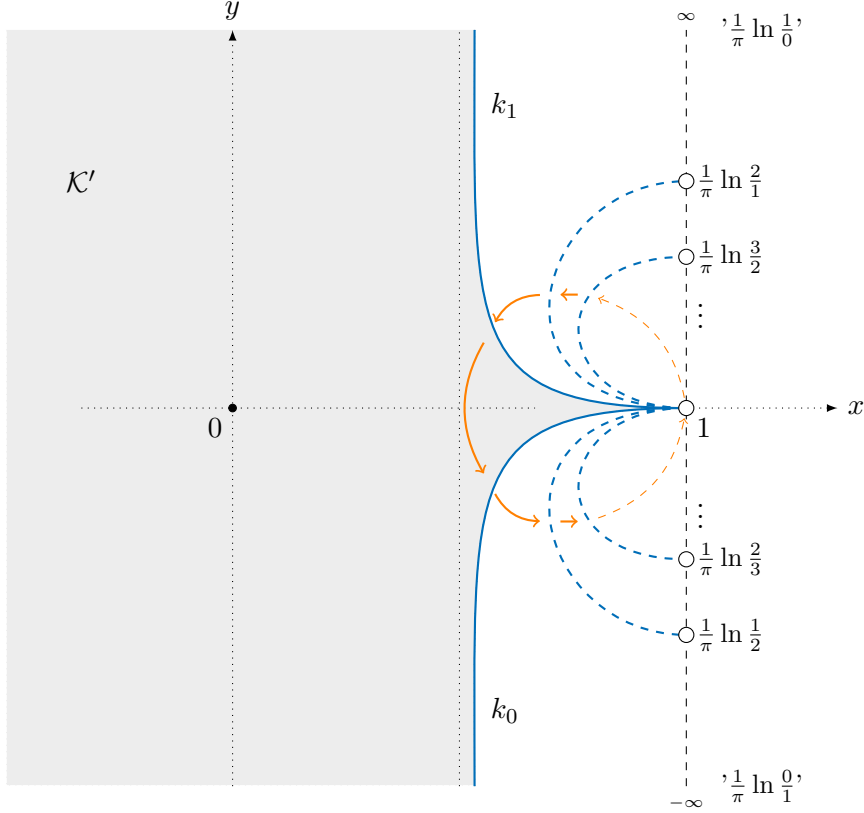
**Theorem 7.5.4.**  *$\text{Stab}(A_2)$  is isomorphic to  $\mathbb{C}^2$  as complex manifold.*

*Proof.* The theorem follows from  $\text{Stab}(A_2)/\mathbb{C} \simeq \mathcal{M}_C \simeq \mathbb{C}$  and  $H^1(\mathbb{C}, \mathcal{O}) = 0$ .  $\square$

### 7.5.2 Comparing to the stability space of Kronecker quiver

Let  $\mathbb{P}^1$  be the projective space of dimension 1 over  $\mathbb{C}$  and  $\text{Coh } \mathbb{P}^1$  the category of coherent sheaves on  $\mathbb{P}^1$ . Notice that we have a derived equivalence

$$\mathcal{D}^b(\text{Coh } \mathbb{P}^1) \cong \mathcal{D}^b(\text{mod } \mathbb{C}K) \tag{7.5.3}$$



**Figure 7-9:** *A fundamental domain for  $(\text{Stab}(\mathbb{P}^1)/\mathbb{C})/\mathbb{Z}$*

sending  $\{\mathcal{O}(-1), \mathcal{O}\}$  to  $\{I_1[-1], P_1\}$ , where  $K$  is the Kronecker quiver (c.f. Section 7.4.5) and  $P_1, I_1$  are the simples in  $\text{mod } \mathbb{C}K$ . We will not distinguish these two derived categories under (7.5.3) and denote it by  $\mathcal{D}(K)$ . Let  $\text{Stab}(\mathbb{P}^1)$  be its stability space.

Figure 7-5 and Figure 7-6 show the quotient graphs  $\text{EG}(A_2)/[1]$  and  $\text{EG}(K)/[1]$  respectively. Thus, roughly speaking, the difference between  $\text{Stab}(A_2)$  and  $\text{Stab}(K)$  is the  $\mathcal{C}_3$ -action and  $\mathbb{Z}$ -action. In this subsection, we will calculate  $\text{Stab}(\mathbb{P}^1)$  using the same method as  $\text{Stab}(A_2)$  and give an alternative interpretation of the proof of [45, Theorem 1.1], which says  $\text{Stab}(\mathbb{P}^1)$  is isomorphic to  $\mathbb{C}^2$  as a complex manifold.

By Theorem 3.1.4, a direct calculation shows that all hearts in  $\mathrm{EG}^\circ(K)$  are

$$\{\mathcal{H}_{i,j,k} \mid \text{Sim } \mathcal{H}_{i,j,k} = \{\mathcal{O}(i)[j], \mathcal{O}(i-1)[j+k]\}, i, j \in \mathbb{Z}, k \in \mathbb{N}\}.$$

Then similar to Lemma 7.5.1, we have the following lemma (for the proof, see, e.g. [45, Lemma 4.2]).

**Lemma 7.5.5.** *Let  $\sigma = (Z, \mathcal{P})$  be a stability condition in  $\mathcal{D}(K)$ . There exists  $\zeta \in \mathbb{C}$  and an positive integer  $m$  such that the simples in the heart of  $\zeta \circ \sigma$  are  $\mathcal{O}$  and  $\mathcal{O}(-i)[m]$ . In particular, up to  $\mathbb{C}$ -action, there are three types of stability conditions on  $\mathcal{D}(K)$ :*

- *Every indecomposable object is stable.*

- Up to shift, two indecomposables are stable and others are semistable (but not stable).
- Up to shift, two indecomposables are stable and others are not semistable.

Similar to  $A_2$  case, we have the following diagram

$$\begin{array}{ccc}
 & \text{Stab}(\mathbb{P}^1) & \\
 \swarrow /_{\text{Aut}} & & \searrow /_{\mathbb{C}} \\
 \mathcal{K}_A & & \mathcal{K}_C \\
 \searrow /_{\mathbb{C}^*} & & \swarrow /_{\mathbb{Z}} \\
 & \mathcal{K} &
 \end{array} \tag{7.5.4}$$

where  $\mathcal{K} = \text{Aut } \mathcal{D}(K) \backslash \text{Stab}(\mathbb{P}^1) / \mathbb{C}$ . Notice that the  $\mathbb{Z}$ -action on  $\mathcal{K}_C$ , denote it by  $\vartheta$ , is induced by the auto-equivalence  $\otimes \mathcal{O}(1)$  on  $D(K)$ . We have a conformal map  $F : R \rightarrow \mathcal{K}_C$ , where  $R$  is as in (7.5.2) such that  $F(\Theta) = [\sigma]$  in  $\mathcal{K}_C$  and the stability condition  $\sigma = (Z, \mathcal{P})$  is determined by the following conditions

- $Z(\mathcal{O}) = 1$  and  $Z(\mathcal{O}(-1)[1]) = \exp(i\pi\Theta)$ ;
- The simples in the heart of  $\sigma$  are  $\mathcal{O}$  and  $\mathcal{O}(-1)[s+1]$ , where  $s = -\lfloor \text{Im } \Theta \rfloor$ .

By Lemma 7.5.5, we have the following decomposition of  $\mathcal{K}_C$  by a direct calculation

$$\mathcal{K}_C = \overline{F(V)} \cup \bigcup_{j \in \mathbb{Z}} \vartheta^j \circ F(V_2)$$

where  $V, V_2$  are as in Section 7.5.1. Let

$$\begin{cases} k_1 = \{\Theta = x + y\mathbf{i} \mid x \in (\frac{1}{2}, 1), y\pi = \ln(-\cos x\pi); \\ k_0 = \{\Theta = x + y\mathbf{i} \mid x \in (\frac{1}{2}, 1), y\pi = -\ln(-\cos x\pi). \end{cases}$$

Then the blue lines in Figure 7-9 is the  $\mathbb{Z}$ -orbit of  $k_1$ . Denote  $\mathcal{K}'$  the region in Figure 7-9 that strictly righted bounded by  $k_1 \cup k_0$ . We have a similar result to Lemma 7.5.2.

**Lemma 7.5.6.** [45, Lemma 4.3]  $\mathcal{K}$  is isomorphic to the surface which is obtained from  $\overline{\mathcal{K}'}$  by identifying the points on the boundary  $k_1 \cup k_0$  with respect to the reflection of  $x$ -axis.

*Proof.* It is equivalent to show that  $\mathcal{K}' \cap V$  is a fundamental domain for the  $\mathbb{Z}$ -action  $\vartheta$  on  $V$ .

There is a conformal isomorphism  $\exp \circ F^{-1} : F(V) \rightarrow \mathbb{C}$ . We consider the induced  $\mathbb{Z}$ -action on  $\mathbb{C}$ . A direct calculation shows that such  $\mathbb{Z}$ -action will identify all  $p_{j+1}/p_j$ 's for  $j \in \mathbb{Z}$ , where  $p_j = 1 + j(Z - 1)$ ,  $j \in \mathbb{Z}$  and some  $Z \in \mathbb{C}$ . Let  $\mathbb{T}_j$  be the triangle with vertex  $(0, p_{j+1}, p_j)$ . Consider all such triangles  $\mathbb{T}_j$ , we have two cases:

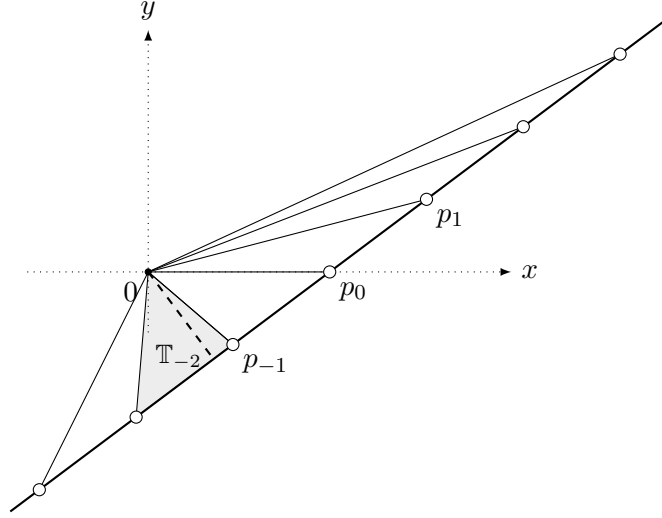


Figure 7-10

- There exists a unique  $k \in \mathbb{Z}$  such that the height from origin in the triangle  $\mathbb{T}_k$  is inside  $\mathbb{T}_k$ . Then  $\mathbb{T}_k$  is the only actual-angled triangle among  $\mathbb{T}_j$ 's. (For example, in Figure 7-10, we have  $k = -2$ .)
- There exists a unique  $k \in \mathbb{Z}$  such that the triangles  $\mathbb{T}_k$  and  $\mathbb{T}_{k-1}$  are right angled while other  $\mathbb{T}_j$ 's are obtuse-angled.

Thus, a fundamental domain for the induced  $\mathbb{Z}$ -action on  $\mathbb{C}$ , consisting of  $p_{k+1}/p_k$  by choosing  $k$  as above. The lemma follows from the following facts

- $\Theta \in \mathcal{K}' \cap V$  if and only if the corresponding triangle  $\mathbb{T}_0$  is actual-angled while other  $\mathbb{T}_j$ 's are obtuse-angled.
- $\Theta \in \partial\mathcal{K}' = k_1 \cup k_0$  if and only if the corresponding triangles  $\mathbb{T}_0$  and  $\mathbb{T}_{-1}$  are right angled while other  $\mathbb{T}_j$ 's are obtuse-angled.

□

## 7.6 Stability Space of Calabi-Yau-N A2-case

### 7.6.1 Autequivalences and the universal cover

Let  $S_1, S_2$  be the simples in the standard heart  $\mathcal{H}_\Gamma$  in  $\mathcal{D}(\Gamma_N A_2)$  such that  $\text{Ext}^!(S_1, S_2) \neq 0$ . Then the braid group  $\text{Br}(\Gamma_N A_2) \cong \text{Br}_3$  has a set of generators  $\phi_{S_1}, \phi_{S_2}$ . By Proposition 6.4.1, we know that  $\xi^3 = [3N - 4]$  generates of the center of  $\text{Br}_3$ , where  $\xi = \phi_{S_2}^{-1} \circ \phi_{S_1}^{-1}$ . Let  $\text{Aut}_0(\Gamma_N A_2)$  be the subgroup of  $\text{Aut } \mathcal{D}(\Gamma_N A_2)$  which is generated

by  $\phi_{S_1}, \phi_{S_2}$  and [1]. Then by Proposition 6.4.1, we have the following commutative diagram of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z}[3N-4] & \longrightarrow & \mathrm{Br}(\Gamma_N A_2) & \longrightarrow & P_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \mathbb{Z}[1] & \longrightarrow & \mathrm{Aut}_0(\Gamma_N A_2) & \longrightarrow & P_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{Z}_{3N-4} & \xlongequal{\quad} & \mathbb{Z}_{3N-4} & & 
 \end{array} \tag{7.6.1}$$

where  $\mathrm{Br}(\Gamma_N A_2) = \mathrm{Br}_3$ ,  $P_2 = \mathrm{PSL}_2(\mathbb{Z})$ , and hence  $\mathrm{Aut}_0(\Gamma_N A_2) \cong \mathrm{Br}_3$ . Therefore we have the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathrm{Stab}^\circ(\Gamma_N A_2) & & \\
 & \swarrow /_{\mathrm{Br}(\Gamma_N A_2)} & & \searrow /_{\mathrm{c}} & \\
 \mathcal{L}_B^N & \xrightarrow{/_{\mathbb{Z}[3N-4]}} & \mathcal{L}_A^N & & \mathcal{L}_C^N \\
 & \searrow /_{\mathrm{c}_B^*} & \searrow /_{\mathrm{c}_A^*} & \swarrow /_{P_2} & \\
 & & \mathcal{L}^N & & 
 \end{array} \tag{7.6.2}$$

Moreover, let  $\Delta = \{\alpha_1, \alpha_2, \alpha_3 \mid \alpha_1 = \alpha_2, \alpha_2 = \alpha_3 \text{ or } \alpha_3 = \alpha_1\}$ ,  $W = \mathfrak{S}_3$  and  $\Delta_0 = \Delta/W$ . We have (c.f. [6])

$$\begin{array}{ccc}
 \mathfrak{h}^{reg} = \left\{ f(x) = \prod (x - \alpha_j) \mid \sum \alpha_j = 0, \alpha_j \in \mathbb{C}, j = 1, 2, 3 \right\} \setminus \Delta & & (7.6.3) \\
 \downarrow W & & \\
 \mathfrak{h}^{reg}/W = \{f(x) = x^3 - a \cdot x + b \mid a, b \in \mathbb{C}\} \setminus \Delta_0 & & 
 \end{array}$$

Write  $\mathfrak{Q} = \mathfrak{h}^{reg}/W$  and denote by  $C^U$  the universal cover of  $\mathfrak{Q}$ . Thus we have the following commutative diagram:

$$\begin{array}{ccc}
 & C^U & \\
 \pi_0 \swarrow & & \searrow /_{\mathrm{c}} \\
 \mathfrak{Q} & & H \\
 \searrow /_{\mathrm{c}^*} & & \swarrow /_{P_2} \\
 & J & 
 \end{array} \tag{7.6.4}$$

where  $J = H/P_2$  is the  $j$ -line. Recall that  $H$  is the upper half plane in  $\mathbb{C}$  and the  $j$ -line is an orbitfold surface with two orbitfold points (of orders 2 and 3).

If  $N = 2$ , we can identify (see [7]) (7.6.4) with the right square of (7.6.2). We will show that this identification works for  $N > 2$  in the following subsection.

### 7.6.2 Deformations

Let  $N \geq 2$ . Let  $\mathcal{N}|_t$  be the area right bounded by  $l_1 \cup l_3$  and left bounded by  $b_t = \{x = -t\}$  (see Figure 7-11). We have the following lemma.

**Lemma 7.6.1.** *The orbitfold  $\mathcal{L}^N$  can be obtained from  $\mathcal{N}|_{(N-2)/2}$  by gluing its boundary  $l_1 \cup l_3 \cup b_{(N-2)/2}$  with respect to the reflection of  $x$ -axis.*

*Proof.* Recall that  $\text{Sim } \mathcal{H}_\Gamma = \{S_1, S_2\}$  with  $\text{Ext}^1(S_2, S_1) \neq 0$ . By Lemma 6.3.1, we have a conformal map

$$\alpha : V \cup \mathcal{N}|_{(N-2)/2} \rightarrow \mathcal{L}^N$$

sending  $\Theta$  to  $[\sigma]$ , and the stability condition  $\sigma = (Z, \mathcal{P})$  is determined by the following conditions

- $Z(S_1) = 1$  and  $Z(S_2) = \exp(i\pi\Theta)$ ;
- The simples in the heart of  $\sigma$  are  $S_1$  and  $S_3[m]$ , where  $m = -[\text{Im } \Theta]$ .

The surjectivity of  $\alpha$  follows by Theorem 5.2.1. To complete the proof, it is essential to show that for  $\Theta_1 \neq \Theta_2 \in V \cup \mathcal{N}|_{(N-2)/2}$  satisfying  $\alpha(\Theta_1) = \alpha(\Theta_2)$ , we have

- 1°. either  $\Theta_1, \Theta_2 \in b_{(N-2)/2}$  such that  $\Theta_1 + \Theta_2 = 2 - N$ .
- 2°. or  $\Theta_1, \Theta_2 \in V$  such that  $\omega_0^k(\Theta_1) = \Theta_2$ , where  $k \in \{\pm 1\}$  and  $\omega_0$  is the  $\mathcal{C}_3$ -action on  $V$  sending  $l_i$  to  $l_{i+1}$ .

Let  $\sigma_1$  and  $\sigma_2$  be the corresponding stability conditions. Notice that for any  $\sigma$  in the orbit of  $\alpha(z)$ , if  $z \in V$ , then there are three (up to shift) indecomposables are semistable; otherwise there are two. Therefore  $\alpha(\Theta_1) = \alpha(\Theta_2)$  implies  $\Theta_i$  are both in  $V$  or neither.

Suppose that  $\Theta_1, \Theta_2 \in \mathcal{N}|_{(N-2)/2} - V$ . Notice that the two stable objects (up to shift) are  $S_1$  and  $S_2$ . Consider the central charges and phases of them with respect to  $\sigma_i$ . Since  $\text{Ext}^1(S_2, S_1) = \text{Ext}^1(S_1, S_2[N-2])$ , either we have

$$\begin{cases} \frac{Z_1(S_1)}{Z_1(S_2)} = \frac{Z_2(S_1)}{Z_2(S_2)}, \\ \varphi_1(S_1) - \varphi_1(S_2) = \varphi_2(S_1) - \varphi_2(S_2). \end{cases} \quad (7.6.5)$$

or

$$\begin{cases} \frac{Z_1(S_1)}{Z_1(S_2)} = \frac{Z_2(S_2[N-2])}{Z_2(S_1)}, \\ \varphi_1(S_1) - \varphi_1(S_2) = \varphi_2(S_2[N-2]) - \varphi_2(S_1). \end{cases} \quad (7.6.6)$$



where  $\phi_i$  is the phase function with respect to  $\sigma_i$ , for  $i = 1, 2$ . Equation (7.6.5) implies  $\sigma_1 = \sigma_2$  which is a contradiction. Hence equation (7.6.6) holds, which implies  $\Theta_1 + \Theta_2 = 2 - N$  as required in 1°.

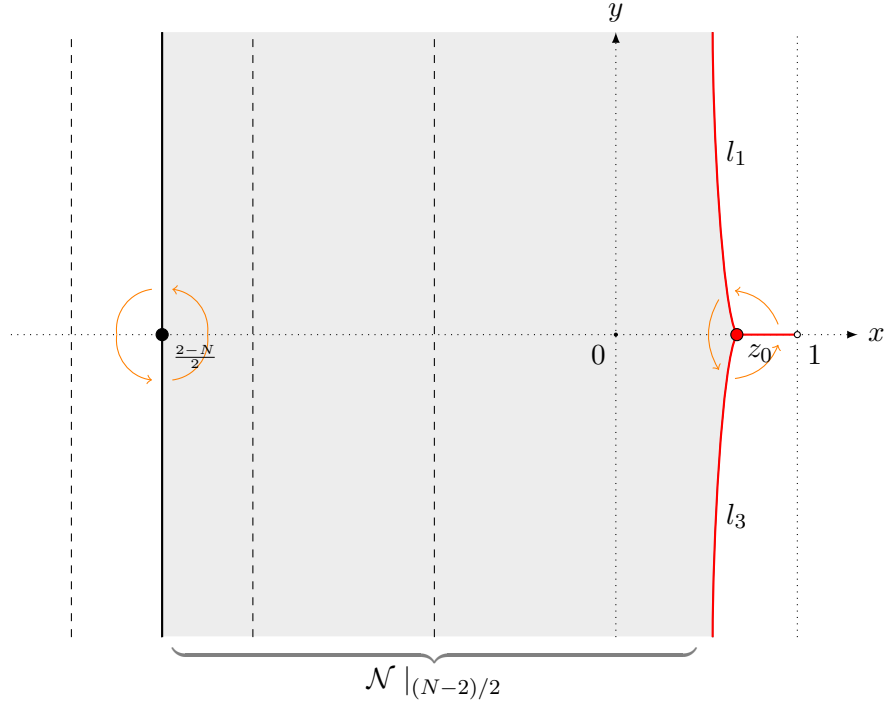
Now let  $\Theta_1, \Theta_2 \in U$ . Then up to shift, there are three semistable objects  $S_1, S_2$  and  $\phi_{S_2}(S_1)$ . Consider their central charges and we know that the triangles  $\mathbb{T}_1$  and  $\mathbb{T}_2$  are similar, where  $\mathbb{T}_i$  has vertices  $0, Z_i(S_1)$  and  $Z_i(S_2)$ . This condition exactly means that  $\sigma_i$  differs by a  $\mathcal{C}_3$ -action (c.f. Section 5) as required in 2°.  $\square$

**Lemma 7.6.2.** *We have a conformal isomorphism  $\mathcal{L}^N \cong \mathcal{L}^2$  for any  $N > 1$ .*

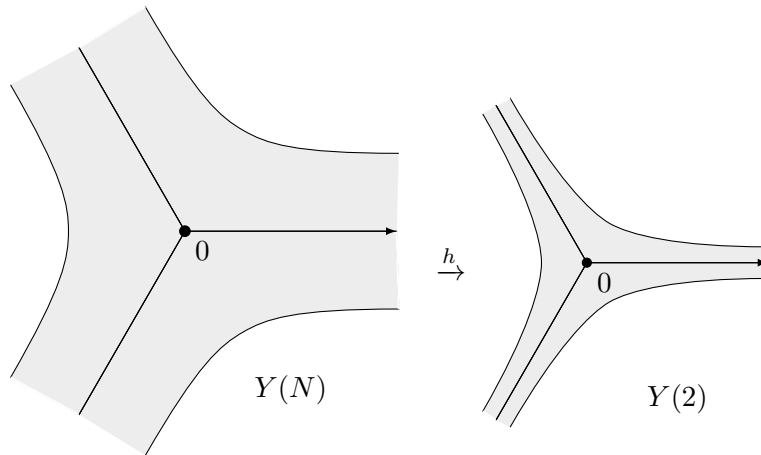
*Proof.* Let  $X(N) = g(\mathcal{N} |_{(N-2)/2})$  where  $g$  is the map in Lemma 7.5.3. We only need to prove that  $X(N)$  is conformally isomorphic to  $X(2)$ . Consider  $Y(N) = \bigcup_{j=1}^3 \omega^j X(N)$ . By Riemann mapping theorem, there is a conformal isomorphism  $h : Y(N) \rightarrow Y(2)$  such that  $h(0) = 0$  and  $h'(0) = 1$  (see Figure 7-12). Let  $h_j = \omega^{-j} \circ h \circ \omega^j$  for any  $j \in \mathbb{Z}$ . Since  $h_j(0) = 0$  and  $h'_j(0) = 1$ , we have  $h = h_j$  by the uniqueness of Riemann mapping theorem. Notice that  $Y(N)$  and  $Y(2)$  are symmetry with respect to  $x$ -axis by construction, hence  $h(l(0)) = l(0)$  by Reflection Principle. Then  $h = h_j$  implies  $h(l(j)) = l(j)$  for any  $j \in \mathbb{Z}$  and hence  $h |_{X(N)} : X(N) \rightarrow X(2)$  is a conformal isomorphism as required.  $\square$

**Theorem 7.6.3.** *We have  $\text{Stab}^\circ(\Gamma_N A_2) \cong \text{Stab}^\circ(\Gamma_2 A_2) \cong C^U$  as complex manifold.*

*Proof.* By Lemma 7.6.2, we have  $\mathcal{L}^N \cong \mathcal{L}^2 \cong J$ . Since the  $\mathbb{C}_A^*$ -bundle  $\mathcal{L}_A^N$  is the principal bundle over  $\mathcal{L}^N$ , we have  $\mathcal{L}_A^N \cong \mathcal{L}_A^2 \cong \mathfrak{Q}$ . Finally, we have  $\text{Aut}_0(\Gamma_N A_2) \cong \text{Aut}_0(\Gamma_x A_2) \cong \text{Br}_3$ , hence  $\text{Stab}^\circ(\Gamma_N A_2)$  and  $\text{Stab}^\circ(\Gamma_2 A_2)$  are both the universal cover of  $\mathfrak{Q}$  which implies the assertion.  $\square$



**Figure 7-11:**  $j$ -line for  $\text{Stab}^\circ(\Gamma_N A_2) / \text{Aut}_0(\Gamma_N A_2)$



**Figure 7-12:** *Deformation*

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